Solution 10

1. Embeddings

a) Find an embedding $S^k \times \mathbb{R}^l \hookrightarrow \mathbb{R}^{k+l}$, where $k, l \geq 1$.

b) Prove that if the m-dimensional manifold M is a product of spheres, then there is an embedding $M \hookrightarrow \mathbb{R}^{m+1}$.

Solution:

a) We consider first the case $l = 1$. Define $\iota \colon S^k \times \mathbb{R} \hookrightarrow \mathbb{R}^{k+1}$, $(x, t) \mapsto xe^t$. It's a homeomorphism onto its image. We claim that it's an embedding.

A smooth atlas on $S^k \subset \mathbb{R}^{k+1}$ can be constructed using the $2k+2$ open hemispheres that cover the sphere in the following way: for $i = 1, \dots, k+1$ let

$$U_i^+ := \{p \in S^k : x^i(p) > 0\}$$
$$U_i^- := \{p \in S^k : x^i(p) < 0\}$$

and define $\varphi_i^\pm \colon U_i^\pm \to U_0(1) \subset \mathbb{R}^k$ by

$$\varphi_i^\pm(x^1, \dots, x^{k+1}) := (x^1, \dots, \widehat{x^i}, \dots, x^{k+1}),$$

where $U_0(1)$ denotes the open unit ball and $\widehat{x^i}$ means that that coordinate is omitted.

For example, given $(x^2, \dots, x^{k+1}) \in \varphi_1^+(U_1^+)$, the inverse is given by

$$(\varphi_1^+)^{-1}(x^2, \dots, x^{k+1}) = \Big(\sqrt{1 - \sum_{i=2}^{k+1} (x^i)^2}, x^2, \dots, x^{k+1}\Big).$$

Notice that $x^i \neq 0$ implies that $\sum_{j \neq i} (x^j)^2 < 1$, and that $\sqrt{\cdot}$ is smooth on $\mathbb{R}_{>0}$.

We endow $S^k \times \mathbb{R}$ with the product atlas $\{(\varphi_i^\pm \times \mathrm{id}_\mathbb{R}, U_i^\pm \times \mathbb{R})\}_i$, then it's a computation to show that ι is smooth. For $(x_0, t_0) \in S^k \times \mathbb{R}$ and φ a chart at (x_0, t_0), the injectivity (in fact, bijectivity since the dimensions coincide) of $d\iota_{x_0,t_0}$ follows from the injectivity of $d(\iota \circ \varphi^{-1})_{\varphi(x_0,t_0)}$ as

$$d\iota_{(x_0,t_0)} = d(\iota \circ \varphi^{-1})_{\varphi(x_0,t_0)} \circ d\varphi_{x_0,t_0}$$

and $d\varphi_{x_0,t_0}$ is a linear isomorphism.

In general $f \colon S^k \times \mathbb{R}^l \hookrightarrow \mathbb{R}^{k+l}$, $(x, t_1, \dots t_l) \mapsto (\iota(x, t_1), t_2 \dots, t_l)$ is an embedding.

b) We proceed by induction on the number of spheres n which form M. The inclusion of the sphere $S^m \subset \mathbb{R}^{m+1}$ is an embedding.

Now let $n \geq 2$ and suppose that $M := \prod_{i=1}^n S^{m_i}$ and $m := \sum_{i=1}^n m_i$ with $m_i \geq 1$. The induction hypothesis gives an embedding $g \colon \prod_{i=2}^n S^{m_i} \hookrightarrow \mathbb{R}^{m-m_1+1}$ and from a) there is and embedding $f \colon S^{m_1} \times \mathbb{R}^{m-m_1+1} \hookrightarrow \mathbb{R}^{m+1}$. As products and compositions of embeddings are embeddings, the map

$$h := f \circ (\mathrm{id}_{S^{m_1}} \times g) \colon M \hookrightarrow \mathbb{R}^{m+1}$$
$$(x_1, \dots x_n) \mapsto f(x_1, g(x_2, \dots, x_n))$$

is also an embedding.

2. The Complex Projective Space

Consider the following equivalence relation on the complex vector space $\mathbb{C}^{n+1}$:

$$x \sim y \quad \Longleftrightarrow \quad x = \lambda y \text{ for some } \lambda \in \mathbb{C} \setminus \{0\}.$$

The quotient space $\mathbb{CP}^n := \big(\mathbb{C}^{n+1} \setminus \{0\}\big)/\sim$, equipped with the quotient topology, is called *complex projective space.*

a) Find a differentiable structure on the topological space $\mathbb{CP}^n$ such that the canonical projection

$$\pi\colon \mathbb{C}^{n+1} \setminus \{0\} \longrightarrow \mathbb{CP}^n$$

is a differentiable map.

b) Prove that S^2 and $\mathbb{CP}^1$ are diffeomorphic.

Solution:

a) We use homogeneous coordinates $[z_0 : \ldots : z_n]$ on $\mathbb{CP}^n$. Define an atlas $\mathcal{A} := \{(U_i, \phi_i)\}_{i=0}^n$ by

$$U_i \coloneqq \{[z_0 : \ldots : z_n] \in \mathbb{CP}^n : z_i \neq 0]\}$$

and $\phi_i\colon U_i \to \mathbb{C}^n$ with

$$\phi_i([z_0 : \ldots : z_n]) \coloneqq \tfrac{1}{z_i}(z_0, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n).$$

The sets U_i's are open and $\mathbb{CP}^n = \bigcup_{i=0}^n U_i$. Moreover the ϕ_i's are homeomorphisms with

$$\phi_i^{-1}(z_1, \ldots, z_n) = [z_1 : \ldots : z_i : 1 : z_{i+1} : \ldots : z_n].$$

The change of coordinates for $U_i \cap U_j$ is given by

$$\phi_i \circ \phi_j^{-1}(z_1, \ldots, z_n) = \left(\tfrac{z_1}{z_i}, \ldots, \tfrac{z_j}{z_i}, \tfrac{1}{z_i}, \tfrac{z_{j+1}}{z_i}, \ldots, \tfrac{z_{i-1}}{z_i}, \tfrac{z_{i+1}}{z_i}, \ldots, \tfrac{z_n}{z_i}\right),$$

which is smooth. Hence the atlas $\mathcal{A}$ defines a C^∞-structure on $\mathbb{CP}^n$.

For the projection it holds that

$$\phi_i \circ \pi(z_0, \ldots, z_n) = \tfrac{1}{z_i}(z_0, \ldots z_{i-1}, z_{i+1}, \ldots, z_n),$$

whenever $\pi(z_0, \ldots, z_n) \in U_i$, and therefore $\pi \in C^\infty$.

b) On S^2 we consider the (surjective) stereographic charts

$$\phi_N\colon S^2 \setminus \{N\} \to \mathbb{R}^2, \qquad \phi_N(x, y, z) = \Big(\frac{x}{1-z}, \frac{y}{1-z}\Big)$$

$$\phi_S\colon S^2 \setminus \{S\} \to \mathbb{R}^2, \qquad \phi_S(x, y, z) = \Big(\frac{x}{1+z}, \frac{y}{1+z}\Big)$$

Notice that

$$\phi_N^{-1}(u, v) = \Big(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\Big).$$

If we consider $S^2 \subset \mathbb{C} \times \mathbb{R}$, then for $z, w \in \mathbb{C}$ and $t \in \mathbb{R}$ we can write

$$\phi_N^{-1}(z) = \Big(\frac{2z}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1}\Big)$$

and

$$\phi_S(w,t) = \frac{w}{1+t}.$$

Define $\Phi\colon \mathbb{CP}^1 \to S^2$ by

$$\Phi([z_0 : z_1]) := \begin{cases} \phi_N^{-1}(\frac{z_0}{z_1}), & \text{if } z_1 \neq 0, \\ N, & \text{if } z_1 = 0. \end{cases}$$

Notice that Φ is a homeomorphism, moreover $\Phi(U_1) = S^2 \setminus \{N\}$ and $\Phi(U_0) = S^2 \setminus \{S\}$. Hence we just need to check the two local expressions

$$\phi_N \circ \Phi \circ \varphi_1^{-1}, \qquad \phi_S \circ \Phi \circ \phi_0^{-1}.$$

In the first case

$$\phi_N \circ \Phi \circ \phi_1^{-1}(z) = \phi_N \circ \Phi([z:1]) = \phi_N \circ \phi_N^{-1}(z) = z,$$

which is smooth with smooth inverse. In the second case $\phi_S \circ \Phi \circ \phi_0^{-1}(z) = \phi_S \circ \Phi([1:z])$.

If $z \neq 0$, then

$$\begin{aligned} \phi_S \circ \Phi([1:z]) &= \phi_S \circ \phi_N^{-1}(\tfrac{1}{z}) \\ &= \phi_S\Big(\frac{2\frac{1}{z}}{|z|^{-2}+1}, \frac{|z|^{-2}-1}{|z|^{-2}+1}\Big) \\ &= \phi_S\Big(\frac{2\bar{z}}{1+|z|^2}, \frac{1-|z|^2}{1+|z|^2}\Big) \\ &= \bar{z}. \end{aligned}$$

If $z = 0$, then $\phi_S \circ \Phi([1:0]) = \phi_S(N) = 0 = \bar{z}$. In both cases $\phi_S \circ \Phi \circ \phi_0^{-1}(z) = \bar{z}$, which is smooth with smooth inverse.

3. Regular Values

Let M and N be manifolds of the same dimension with M compact and let $f\colon M \to N$ be a smooth map. Let $y \in N$ be a regular value of f. Prove the following statements.

a) The preimage $f^{-1}(y)$ has only finitely many elements.

b) The number of elements in the fiber over y is locally constant in N. That is, for every regular value $y \in N$ there exists a neighborhood V of y, such that all $y' \in V$ are regular values and $\#f^{-1}(y) = \#f^{-1}(y')$.

c) If the space of regular values is connected, then $\#f^{-1}(y)$ is constant for all regular values.

Solution:

a) As y is a regular value of f, we know that df_x is surjective for all $x \in f^{-1}(y)$ and since M and N have the same dimension df_x is bijective. Therefore f is locally a diffeomorphism, that is, there exist an open neighborhood U_x of x and an open neighborhood V_x of y such that $f|_{U_x}\colon U_x \to V_x$ is a diffeomorphism. In particular $U_x \cap f^{-1}(y) = \{x\}$.

Moreover f is continuous and $\{y\}$ is closed in N, so $f^{-1}(y) \subset M$ is closed and hence compact. This implies that the open cover $\{U_x\}_{x \in f^{-1}(y)}$ of $f^{-1}(y)$ admits a finite subcover $\{U_{x_i}\}_{i=1}^n$. Together with the above observation we conclude that

$$f^{-1}(y) = f^{-1}(y) \cap \bigcup_{i=1}^n U_{x_i} = \bigcup_{i=1}^n (U_{x_i} \cap f^{-1}(y)) = \bigcup_{i=1}^n \{x_i\} = \{x_1, \dots, x_n\}$$

and hence $f^{-1}(y)$ is finite.

b) Let $f^{-1}(y) = \{x_1, \dots, x_n\}$ with $U_i \ni x_i$ open and $U_i \cap U_j = \emptyset$, such that $f|_{U_i} \colon U_i \to V_i$ is a diffeomorphism (as above, restricting the neighborhoods to make them pairwise disjoint, if necessary).

The set $A \coloneqq M \setminus \bigcup_{i=1}^n U_i$ is closed and hence compact. It follows that $f(A)$ is compact and thus closed. Hence its complement $W \coloneqq N \setminus f(A)$ is open and $y \in W$, since $A \cap f^{-1}(y) = \emptyset$.

We define $V \coloneqq \bigcap_{i=1}^n V_i \cap W$, which is an open neighborhood of y with the additional property that for all $y' \in V$ the preimage is contained in $\bigcup_{i=1}^n U_i$. Therefore it holds that

$$f^{-1}(y') = f^{-1}(y') \cap \bigcup_{i=1}^n U_i = \bigcup_{i=1}^n (U_i \cap f^{-1}(y')) = \bigcup_{i=1}^n \{x'_i\} = \{x'_1, \dots, x'_n\},$$

so $\#f^{-1}(y) = \#f^{-1}(y')$. Moreover all the x'_i (and hence the y') are regular, as $f|_{U_i} \colon U_i \to V_i$ is a diffeomorphism.

c) Let R be the set of all regular values of f. From b) it follows that the map $g \colon R \to \mathbb{Z}$, $y \mapsto \#f^{-1}(y)$ is continuous. Thus $g(R)$ is connected, that is, g is constant on R.