Solution 11

1. Ideal Triangles in the Hyperbolic Plane

Let $H^2 := \{x+iy \in \mathbb{C} \colon y > 0\}$ be the upper half-plane endowed with the hyperbolic metric $(g_{ij})(x,y) = \frac{1}{y^2}(\delta_{ij})$. The "point at infinity", ∞, denotes the "point" which corresponds to $\lim_{y\to\infty}(x,y)$ for all $x \in \mathbb{R}$. An *ideal triangle* is a geodesic triangle whose all vertices lie on the x-axis or whose two vertices lie on the x-axis and one at the point at infinity.

a) Prove that every ideal triangle is congruent to the ideal triangle with vertices $A = (0,0)$, $B = (1,0)$ and $C = \infty$.

b) Prove that the area of any ideal triangle is π.

Solution:

a) Let U, V, W be the vertices of an ideal triangle.
If $U, V, W \in \mathbb{R}$ with $U < V < W$, consider

$$f(z) := \frac{(V-W)(z-U)}{(V-U)(z-W)} \in \operatorname{Isom}(H^2),$$

otherwise if $U < V \in \mathbb{R}, W = \infty$, consider

$$f(z) := \frac{(z-U)}{(V-U)} \in \operatorname{Isom}(H^2).$$

Then $f(U) = A$, $f(V) = B$ and $f(W) = C$.

b) From a) it suffices to show this result for a specific ideal triangle Δ. We choose $A = -1, B = 1, C = \infty$. Then

$$F = \int_\Delta \sqrt{\det g}\, dA = \int_{-1}^{1}\int_{\sqrt{1-x^2}}^{\infty} \tfrac{1}{y^2}\, dy\, dx = \int_{-1}^{1} \left[-\tfrac{1}{y}\right]_{\sqrt{1-x^2}}^{\infty} dx$$
$$= \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\, dx = [\arcsin x]_{-1}^{1} = \tfrac{\pi}{2} - (-\tfrac{\pi}{2}) = \pi.$$

2. Hopf Fibration

Let $\pi \colon \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$ be the canonical projection from Ex. 2 of Exercise Sheet 10. The *Hopf fibration*

$$H \colon S^{2n+1} \to \mathbb{CP}^n$$

is given by the restriction of π to $S^{2n+1} \subset \mathbb{C}^{n+1} \setminus \{0\}$.

a) Let $n = 1$. Describe the fibers of H over a point $x \in \mathbb{CP}^1$, that is, $H^{-1}(x)$.

b) Prove that $H \colon S^{2n+1} \to \mathbb{CP}^n$ is a submersion.

Solution:

a) For $z, z' \in H^{-1}(x)$ we have

$$H(z) = H(z') \Leftrightarrow z \sim z' \Leftrightarrow \exists \lambda \in \mathbb{C} : z = \lambda z'.$$

Since $z, z' \in S^3$, it follows that $|\lambda| = |\lambda||z'| = |z| = 1$. Thus

$$H^{-1}(x) = \{\lambda z : \lambda \in S^1\} \cong S^1.$$

b) It suffices to check the surjectivity of dH_p for $p = (1, 0, \ldots, 0)$[1]. For $i = 1, \ldots, n$ and $\lambda \in S^1 \subset \mathbb{C}$ define $\gamma_i \colon (-\varepsilon, \varepsilon) \to S^{2n+1}$ by

$$\gamma_i(t) := (\cos t, \underbrace{0, \ldots, 0}_{i-1}, \lambda \sin t, 0, \ldots, 0) \in S^{2n+1} \subset \mathbb{C}^{n+1}$$

(that is $z_0 = \cos t$, $z_i = \lambda \sin t$). Then

$$\frac{d}{dt}\Big|_{t=0} (\phi_0 \circ H \circ \gamma_i)(t) = \frac{d}{dt}\Big|_{t=0} (\underbrace{0, \ldots, 0}_{i-1}, \lambda \tfrac{\sin t}{\cos t}, 0, \ldots, 0) = \tfrac{\lambda}{\cos^2 t}\Big|_{t=0} \cdot e_i = \lambda \cdot e_i,$$

see Ex. 2 of Exercise solution 10 for the definition of ϕ_0. So we conclude that $d(\phi_0 \circ H)_p(TS^{2n+1}_p) = \mathbb{C}^n = d(\phi_0)_{H(p)}(T\mathbb{CP}^n_{H(p)})$ and therefore dH_p is surjective.

[1] For any $q \in S^{2n+1} \subset \mathbb{C}^{n+1}$, there exists $A \in \mathrm{U}(n+1)$ such that $A(p) = q$. A defines a diffeomorphism on $\mathbb{CP}^n$ (with A^* as the inverse map) satisfying $H \circ A = A \circ H$, so we have $dH_q \circ dA_p = dA_{H(p)} \circ dH_p$ and obtain surjectivity of dH_q from that of dH_p.

3. Mapping Degree of Gauss Map

Let $M \subset \mathbb{R}^3$ be a compact, connected surface (without boundary) with exterior Gauss map $N \colon M \to S^2$. Prove that

$$\deg(N) = \frac{1}{2}\chi(M).$$

Hint: Use Exercise 3 of Sheet 7.

Solution:

Note that $p \in M$ is a regular point of N if and only if $K(p) \neq 0$, since $K(p) = \det(-dN_p)$. Moreover

$$\operatorname{sgn}(dN_p) = \begin{cases} +1, & K(p) > 0, \\ -1, & K(p) < 0. \end{cases}$$

We define $M_+ := \{p \in M : K(p) > 0\}$ and $M_- := \{p \in M : K(p) < 0\}$.

By the Theorem of Gauss-Bonnet and Exercise 3 of Sheet 7 we obtain

$$2\pi\chi(M) = \int_M K\, dA = \int_{M_+} |K|\, dA - \int_{M_-} |K|\, dA = A(N|_{M_+}) - A(N|_{M_-}).$$

Now, let $R \subset S^2$ be set of all regular values of N.

The area of N is counted with multiplicities and from Sard's Theorem almost every value of N is regular, hence we compute

$$\begin{aligned}
A(N|_{M_+}) - A(N|_{M_-}) &= \int_{N(M_+)} \#N|_{M_+}^{-1}(q)\, dA(q) - \int_{N(M_-)} \#N|_{M_-}^{-1}(q)\, dA(q) \\
&= \int_{N(M_+)\cap R} \#N|_{M_+}^{-1}(q)\, dA(q) - \int_{N(M_-)\cap R} \#N|_{M_-}^{-1}(q)\, dA(q) \\
&= \int_R \underbrace{\left(\#N|_{M_+}^{-1}(q) - \#N|_{M_-}^{-1}(q)\right)}_{=\deg N} dA(q) \\
&= A(S^2)\deg N = 4\pi \deg N,
\end{aligned}$$

hence $2\pi\chi(M) = 4\pi \deg N$ and therefore $\deg(N) = \frac{1}{2}\chi(M)$.