Solution 13

1. Pullback

Let M, N be smooth manifolds and $F \colon N \to M$ be a smooth map. For $\omega \in \Omega^s(M)$ and $\theta \in \Omega^t(M)$ prove that

a) $F^*(\omega \wedge \theta) = F^*\omega \wedge F^*\theta$,

b) $F^*(d\omega) = d(F^*\omega)$.

Solution:

a) Let $p \in M$ and $v_1 \dots, v_{s+t} \in TM_p$, then

$$\begin{aligned}
&(F^*(\omega \wedge \theta))_p(v_1, \dots, v_{s+t}) \\
&\quad = (\omega \wedge \theta)_{F(p)}(dF_p(v_1), \dots, dF_p(v_{s+t})) \\
&\quad = \sum_{\sigma \in S_{s,t}} \operatorname{sgn}(\sigma) \cdot \omega_{F(p)}(dF_p(v_{\sigma(1)}), \dots, dF_p(v_{\sigma(s)})) \cdot \theta_{F(p)}(dF_p(v_{\sigma(s+1)}), \dots, dF_p(v_{\sigma(s+t)})) \\
&\quad = \sum_{\sigma \in S_{s,t}} \operatorname{sgn}(\sigma) \cdot (F^*\omega)_p(v_{\sigma(1)}, \dots, v_{\sigma(s)}) \cdot (F^*\theta)_p(v_{\sigma(s+1)}, \dots, v_{\sigma(s+t)}) \\
&\quad = (F^*\omega \wedge F^*\theta)_p(v_1, \dots, v_{s+t}),
\end{aligned}$$

where $S_{s,t} := \{\sigma \in S_{s+t} : \sigma(1) < \dots < \sigma(s), \sigma(s+1) < \dots < \sigma(s+t)\}$. As $p \in M$ and $v_1, \dots, v_{s+t} \in TM_p$ were arbitrary, the assertion follows.

b) Let (φ, U) be a chart of M. It suffices to prove the formula for $\omega = f d\varphi^{i_1} \wedge \ldots \wedge d\varphi^{i_s}$. We proceed by induction on s. If $s = 0$, then $\omega = f$ and for $p \in F^{-1}(U)$ and $X \in TN_p$:

$$\begin{aligned}
(F^*(df))_p(X) &= df_{F(p)}(dF_p(X)) \\
&= d(f \circ F)_p(X) \\
&= d(F^*f)_p(X),
\end{aligned}$$

where we have used the definition of F^* in the first and third equalities, and the chain rule in the second.

Now let $s \geq 1$ and suppose that the formula holds for $s-1$, then

$$\omega = f \underbrace{d\varphi^{i_1} \wedge \ldots \wedge d\varphi^{i_{s-1}}}_{:= d\varphi^I} \wedge d\varphi^{i_s} = f\, d\varphi^I \wedge d\varphi^{i_s},$$

and

$$\begin{aligned}
d(F^*\omega) &\overset{(a)}{=} d\big(F^*(f\, d\varphi^I) \wedge F^*(d\varphi^{i_s})\big) \\
&= d\big(F^*(f\, d\varphi^I)\big) \wedge F^*(d\varphi^{i_s}) + (-1)^{s-1} F^*(f\, d\varphi^I) \wedge d\big(F^*(d\varphi^{i_s})\big) \\
&= d\big(F^*(f\, d\varphi^I)\big) \wedge F^*(d\varphi^{i_s}) \\
&= F^*\big(d(f\, d\varphi^I)\big) \wedge F^*(d\varphi^{i_s}) \\
&= F^*\big(df \wedge d\varphi^I)\big) \wedge F^*(d\varphi^{i_s}) \\
&\overset{(a)}{=} F^*\big(df \wedge d\varphi^I \wedge d\varphi^{i_s}\big) \\
&= F^*(d\omega),
\end{aligned}$$

where in the third equality we have used that $d\big(F^*(d\varphi^{i_s})\big) = d\big(d(F^*\varphi^{i_s})\big) = 0$ and in the fourth the induction step.

2. Volume Forms

Let M be an m-dimensional manifold. A *volume form* ω on M is a nowhere vanishing m-form, that is, $\omega_p \neq 0$ $(\in \Lambda_m(TM_p^*))$ for all $p \in M$.
Prove that there is a volume form on M if and only if M is orientable.

Solution:

Suppose that M is orientable. Let $\{(\varphi_\alpha, U_\alpha)\}_{\alpha\in A}$ be a locally-finite oriented atlas and $\{\tau_\alpha\}_{\alpha\in A}$ a subordinate partition of unity.

For all $\alpha \in A$ define

$$\omega_\alpha := \begin{cases} \tau_\alpha\, d\varphi_\alpha^1 \wedge \ldots \wedge d\varphi_\alpha^m & \text{on } U_\alpha \\ 0 & \text{otherwise.} \end{cases}$$

and set $\omega := \sum_{\alpha\in A} \omega_\alpha \in \Omega^m(M)$. For all $p \in M$ there exists α with $\tau_\alpha(p) > 0$ and also

$$(\omega_\beta)_p\Big(\frac{\partial}{\partial\varphi_\alpha^1}\Big|_p, \ldots, \frac{\partial}{\partial\varphi_\alpha^m}\Big|_p\Big) \begin{cases} = \tau_\alpha(p), & \beta = \alpha \\ \geq 0, & \beta \neq \alpha, \end{cases}$$

since $\{(\varphi_\alpha, U_\alpha)\}_{\alpha\in A}$ is an oriented atlas. Therefore

$$\omega_p\Big(\frac{\partial}{\partial\varphi_\alpha^1}\Big|_p, \ldots, \frac{\partial}{\partial\varphi_\alpha^m}\Big|_p\Big) \geq \tau_\alpha(p) > 0$$

and ω is a volume form.

For the other implication let ω be a volume form on M and $\{(\varphi_\alpha, U_\alpha)\}_{\alpha\in A}$ an atlas on M. Notice that for each $\alpha \in A$,

$$(\varphi_\alpha^{-1})^*(\omega|_{U_\alpha}) = f_\alpha \underbrace{dx^1 \wedge \ldots \wedge dx^m}_{=:\omega_0}$$

for $f_\alpha \in C^\infty(\varphi_\alpha(U_\alpha))$ nowhere vanishing (beacuse ω is a volume form).

We can modify the atlas $\{(\varphi_\alpha, U_\alpha)\}_{\alpha\in A}$ such that $f_\alpha > 0$ for all $\alpha \in A$.

For $\alpha, \beta \in A$ consider the change of coodinates $H := \varphi_\beta \circ \varphi_\alpha^{-1}$, then on $\varphi_\alpha(U_\alpha \cap U_\beta)$ it holds that

$$f_\alpha\omega_0 = H^*(f_\beta\omega_0) = (f_\beta \circ H) \cdot H^*\omega_0 = (f_\beta \circ H) \cdot \det J_H \cdot \omega_0.$$

Since $f_\alpha > 0$ and $f_\beta \circ H > 0$ we conclude that $\det J_H > 0$ and thus $\{(\varphi_\alpha, U_\alpha)\}_{\alpha\in A}$ is an oriented atlas.

3.

Let M and N be two compact, oriented smooth manifolds of dimension m. Show that if $F, G\colon M \to N$ are smoothly homotopic maps and $\omega \in \Omega^m(N)$ is an m-form on N, then

$$\int_M F^*\omega = \int_M G^*\omega.$$

Solution:

Let $H\colon M \times [0,1] \to N$ be a smooth homotopy from F to G. By the theorem of Stokes, and since $d\omega \in \Omega^{m+1}(N) = \{0\}$ (as $\dim(N) = m$),

$$\int_{\partial(M\times[0,1])} H^*\omega = \int_{M\times[0,1]} d(H^*\omega) = \int_{M\times[0,1]} H^*(d\omega) = 0.$$

Consider the inclusion maps

$$i_0\colon M \to M \times \{0\} \subset \partial(M \times [0,1]),$$
$$i_1\colon M \to M \times \{1\} \subset \partial(M \times [0,1]),$$

and note that $F^*\omega = (H \circ i_0)^*\omega = i_0^*(H^*\omega)$ and $G^*\omega = i_1^*(H^*\omega)$. Regardless of how $M \times [0,1]$ is oriented, exactly one of the maps i_0, i_1 is orientation preserving with respect to the induced orientation on $\partial(M \times [0,1])$. We can assume that i_1 preserves orientation. Then

$$\int_M G^*\omega - \int_M F^*\omega = \int_{M\times\{1\}} H^*\omega + \int_{M\times\{0\}} H^*\omega = 0.$$

Alternative. One can also use the degree formula mentioned just before Theorem 11.6 (Stokes) in the lecture notes. For this, however, one has to revert to the case of connected target manifolds (otherwise the mapping degree is not defined). If H is a homotopy as above, and M' is a connected component of M, then the restriction of H to $M' \times [0,1]$ is a homotopy from $F|_{M'}$ to $G|_{M'}$ with values in a connected component N' of N. Then

$$\int_{M'} (F|_{M'})^*\omega = \deg(F|_{M'}) \int_{N'} \omega = \deg(G|_{M'}) \int_{N'} \omega = \int_{M'} (G|_{M'})^*\omega,$$

and the general result is obtained by summation over the (finitely many) connected components of M.