Solution 2

1. Characterization of convex curves

Let $c \in C^2([0,L],\mathbb{R}^2)$ be a simple C^2-closed curve parametrized by arc-length. Show that the following two statements are equivalent:

(i) The curvature κ_{or} of c doesn't change sign, that is, $\kappa_{\text{or}}(t) \geq 0$ for all $t \in [0,L]$ or $\kappa_{\text{or}}(t) \leq 0$ for all $t \in [0,L]$.

(ii) The curve c is *convex*, that is, the image of c is the boundary of a convex subset $C \subset \mathbb{R}^2$.

Solution:

We begin by noticing that c is convex if and only if for each $t \in [0,L]$ the curve lies in one of the closed half-planes determined by the tangent line at $c(t)$.

Let $\dot{c} = e_1 : [0,L] \to S^1$ be the tangent indicatrix and let $\theta : [0,L] \to \mathbb{R}$ be a continuous (hence differentiable, as seen in class) polar angle function for e_1, that is,

$$e_1(s) = (\cos\theta(s), \sin\theta(s))$$

for all $s \in [0,L]$. Then

$$\dot{e}_1(s) = \theta'(s)(-\sin\theta(s), \cos\theta(s)) = \theta'(s)e_2(s),$$

and using the first Frenet equation we conclude that $\theta' = \kappa_{\text{or}}$, thus

$$\int_0^t \kappa_{\text{or}}(s)\,\mathrm{d}s = \theta(t) - \theta(0).$$

This shows that the condition that κ_{or} doesn't change sign is equivalent to θ being monotonic.

We now prove $(i) \Rightarrow (ii)$. Suppose that κ_{or} doesn't change sign. Without loss of generality we might assume that it's always ≥ 0 and θ is non-decreasing. By Jordan curve theorem (see e.g. Allen Hatcher's "Algebraic Topology" 2.B), $\mathbb{C} \setminus c([0,L])$ consists of exactly two open connected components. Let U be the bounded component, then the Jordan curve theorem also says $\partial U = c[0,L]$. Our purpose is to show that U is convex. Pick any $t_0 \in [0,L]$, denote by $n_0 := e_2(t_0)$ the normal vector to c at t_0 such that $(e_1(t_0), n_0)$ is positively oriented and define $h : [0,L] \to \mathbb{R}$ by

$$h(t) := \langle c(t) - c(t_0), n_0\rangle.$$

Lemma: $h(t)$ doesn't change sign for $t \in [0,L]$.
Proof of lemma: Let T be the tangent line to c at $c(t_0)$, the map h measures the distance of $c(t)$ from T. Suppose h changes signs on $[0,L]$ (i.e. c runs on both sides of T), then since $[0,L]$ is compact and c is C^2-closed (in particular, h can be extended to an L-periodic C^2-function on $\mathbb{R}$), the map h has a maximum at t_1 and a minimum at t_2 with $h'(t_1) = h'(t_2) = 0$, $h(t_2) < h(t_0) = 0 < h(t_1)$.

Therefore $\langle \dot{c}(t_0), n_0\rangle = \langle \dot{c}(t_1), n_0\rangle = \langle \dot{c}(t_2), n_0\rangle = 0$, that is, $\dot{c}(t_0), \dot{c}(t_1), \dot{c}(t_2)$ are parallel and so there are $s_1 < s_2 \in \{t_0, t_1, t_2\}$ such that $\dot{c}(s_1) = \dot{c}(s_2)$.

Since θ is non-decreasing this implies $\theta(s_2) - \theta(s_1) = 2\pi k$ for some $k \in \mathbb{N}_0$. The theorem of turning tangents (Hopf Umlaufsatz), together with the fact that θ is non-decreasing, implies $0 \leq \theta(s_2) - \theta(s_1) \leq 2\pi$, hence k is either 0 or 1.

If $k = 0$ then θ is constant on $[s_1, s_2]$, which means that $\dot{c}(t) = \dot{c}(s_1)$ is orthogonal to n_0 for $t \in [s_1, s_2]$. Hence $\dot{h} = 0$ on $[s_1, s_2]$ and $h(s_1) = h(s_2)$, contradiction arises.

If $k = 1$ then by Hopf Umlaufsatz θ must be constant on $[0, s_1]$ and on $[s_2, L]$, which by the same argument implies that $\dot{h} = 0$ on $[0, s_1] \cup [s_2, L]$. Hence $h(s_2) = h(L) = h(0) = h(s_1)$. Again this is a contradiction. The lemma is proved. $\square$

As a corollary of the above lemma, for any $t_0 \in [0, L]$, $\bar{U}$ is also contained in one side of the tangent line T at $c(t_0)$ (assume $h \geq 0$ on $[0, L]$, then the connected set $\{v \in \mathbb{R}^2, \langle v - c(t_0), n_0 \rangle < 0\}$ is contained in $\mathbb{R}^2 \backslash c([0, L])$, hence in the unbounded connected component of $\mathbb{R}^2 \backslash c([0, L])$. This implies $U \subset \{v \in \mathbb{R}^2 : \langle v - c(t_0), n_0 \rangle \geq 0\}$). Suppose U is not convex, then there exists $p, q \in U$ such that the line segment connecting p and q is not contained in U. Hence there exists $s \in (0, 1)$ such that $sp + (1 - s)q \in \partial U = c([0, L])$. We write $sp + (1 - s)q = c(s_0)$ for $s_0 \in [0, L]$. If $p - q$ is not parallel to $\dot{c}(s_0)$, then $p, q \in U$ lie on different sides of the tangent line T at $c(s_0)$, contradiction arises. If $p - q$ is parallel to $\dot{c}(s_0)$, then since U is open, there exists an open neighborhood of p contained in U, hence we also obtain points in U lying on different sides of T, contradiction arises. Therefore, U is open with $\partial U = c([0, L])$.

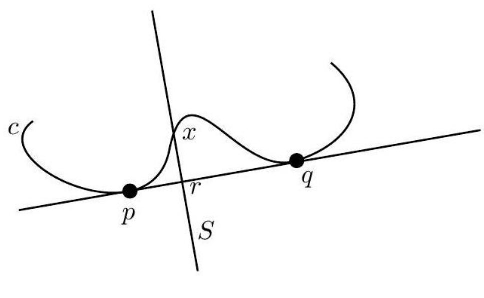
We now prove $(ii) \Rightarrow (i)$. Suppose $c([0, L]) = \partial V$ for a convex set V.

Lemma: For $t_0 \in [0, L]$, define $n_0 := e_2(t_0)$, $h(t) := \langle n_0, c(t) - c(t_0) \rangle$ as above, where (e_1, e_2) is the Frenet frame of c. Then h doesn't change sign on $[0, L]$.

Proof of lemma: by the supporting hyperplane theorem (see e.g. "Convex Optimization" by Boyd & Vandenberghe, or the more general version Hahn-Banach separation theorem), there exists $n_1 \in \mathbb{R}^2, |n_1| = 1$ such that $\langle n_1, v - c(t_0) \rangle \geq 0$ for any $v \in V$, hence for any $v \in \partial V = c([0, L])$ by continuity. It follows that the function $h_1(t) := \langle n_1, c(t) \rangle$ achieves a minimum at t_0. Since h_1 can be extended to an L-periodic C^2 function on $\mathbb{R}$, we have $h_1'(t_0) = 0$, which means $\langle n_1, \dot{c}(t_0) \rangle = 0$. So $n_1 = \pm n_0$, the lemma is proved. $\square$

Suppose θ is not monotonic on $[0, L]$, then there are $t_1 < t_2$ in $[0, L]$ with $\theta(t_1) = \theta(t_2)$ and θ not constant on $[t_1, t_2]$. By Hopf Umlaufsatz, $\dot{c}$ maps surjectively onto S^1, so there exists $t_3 \in [0, L]$ with $\dot{c}(t_1) = -\dot{c}(t_3)$. If the tangent lines at $c(t_1), c(t_2)$ and $c(t_3)$ are pairwise distinct (i.e. $\langle e_2(t_1), c(t_i) \rangle$ for $i \in \{1, 2, 3\}$ are pairwise distinct), then they are parallel and one of them lies between the other two. This can't be the case by the lemma above, thus two of the tangent lines coincide and there are points $p, q \in \{c(t_1), c(t_2), c(t_3)\}$ lying on the same tangent line. Write $p = c(s_1)$, $q = c(s_2)$, $s_1 < s_2$.

Let $\overline{pq}$ denote the closed straight line segment from p to q. We claim that $\overline{pq} \subset c([0, L])$. Suppose that $r \in \overline{pq}$ is not on c and denote by S the straight line perpendicular to $\overline{pq}$ at r.



Since p and q lie on distinct sides of S and c is simple closed, S intersect c in at least two points, say x and y (S intersects c on $c\big((s_1, s_2)\big)$ and $c\big((s_2, L] \cup [0, s_1)\big)$ respectively) where x is the nearest point to r. Then x is between r and y since the lemma above implies x, y lie on the same side of the straight line passing through p, q, and we have x, y, r, p, q are pairwise distinct. Thus x is in the interior of the convex hull H of p, q, y ($H := \{\lambda_1 p + \lambda_2 q + \lambda_3 y : \lambda_1, \lambda_2, \lambda_3 \geq 0, \lambda_1 + \lambda_2 + \lambda_3 = 1\}$). By the lemma above, there exists $n_x \in \mathbb{R}^2 \setminus \{0\}$ such that $\langle n_x, c(t) - x \rangle \geq 0$ for any $t \in [0, L]$, in particular, $\langle n_x, p - x \rangle$, $\langle n_x, q - x \rangle$, $\langle n_x, y - x \rangle \geq 0$, hence $\langle n_x, z - x \rangle \geq 0$ for any $z \in H$. But x is in the interior of H, so there exists small

$\lambda > 0$ s.t. $x - \lambda n_x \in H$ but $\langle n_x, x - \lambda n_x - x\rangle < 0$, contradiction arises.

Hence $\overline{pq} \subset c([0, L])$. Let $I := \overline{pq} \setminus \{p, q\}$. Note $c([s_1, s_2])$ is compact, hence $c([s_1, s_2]) \cap I$ is closed in I. Similarly, $c([0, s_1] \cup [s_2, L])$ is closed in I. Since I is connected and is the union of two disjoint closed subsets $c([s_1, s_2]) \cap I$ and $c([0, s_1] \cup [s_2, L])$, we have I equals to one of them. Hence $\overline{pq}$ is contained either in $c([s_1, s_2])$ or $c([0, s_1] \cup [s_2, L])$.

Case 1: $\overline{pq} \subset c([s_1, s_2])$. Suppose there exists $t \in [s_1, s_2]$ such that $c(t) \notin \overline{pq}$, then the connected set $\overline{pq}$ is the union of two disjoint closed subsets $c([s_1, t]) \cap \overline{pq}$ and $c([t, s_2]) \cap \overline{pq}$, in particular, $\{p, q\} \subset c([s_1, t])$ or $\{p, q\} \subset c([t, s_2])$. Since c is simple, we have $t = s_1$ or s_2, which means $c(t) = p$ or q, contradiction arises. Therefore, $c([s_1, s_2]) = \overline{pq}$ and c is the straight line segment from p to q on $[s_1, s_2]$. It follows that $\dot{c}(s_1) = \dot{c}(s_2)$, hence $s_1 = t_1$, $s_2 = t_2$ and θ is a constant on $[t_1, t_2]$ by unique lifting property of covering space, contradiction arises.

Case 2: $\overline{pq} \subset c([0, s_1] \cup [s_2, L])$. By the same argument as above, we have c is a straight line segment on $[0, s_1]$ and $[s_2, L]$ respectively. Hence $\theta(s_2) = \theta(L)$ and $\theta(0) = \theta(s_1)$, and we also have $\dot{c}(s_1) = \dot{c}(0) = \dot{c}(L) = \dot{c}(s_2)$. It follows that $s_1 = t_1, s_2 = t_2$. But we have $\theta(t_1) = \theta(t_2)$ by assumption at the beginning, so $\theta(L) = \theta(s_2) = \theta(s_1) = \theta(0)$, and this is impossible by Hopf Umlaufsatz. Therefore, we proved that θ is monotonic on $[0, L]$, i.e. κ_{or} doesn't change sign.

(Remark: the above proof is still valid if c is only assumed to be C^1-closed, with the condition (i) changed to "θ is monotonic on $[0, L]$".)

2. Submanifolds

Prove that the following matrix groups are submanifolds of $\mathbb{R}^{n\times n}$ and compute their dimensions:

(i) $\mathrm{SL}(n, \mathbb{R}) := \{A \in \mathbb{R}^{n\times n} : \det A = 1\}$,

(ii) $\mathrm{SO}(n, \mathbb{R}) := \{A \in \mathrm{GL}(n, \mathbb{R}) : A^{-1} = A^{\mathrm{T}}, \det A = 1\}$.

Solution:

The idea is to write $\mathrm{SL}(n, \mathbb{R})$ and $\mathrm{SO}(n, \mathbb{R})$ as preimages of regular values of smooth maps and apply regular value theorem.

(i) Consider the smooth map $F : \mathbb{R}^{n\times n} \to \mathbb{R}, A \mapsto \det A$. Notice that F is smooth and $\mathrm{SL}(n, \mathbb{R}) = F^{-1}(1)$ so we want to show that 1 is a regular value for F, that is, $D_A F : \mathbb{R}^{n\times n} \longrightarrow \mathbb{R}$ is surjective for all A in $\mathrm{SL}(n, \mathbb{R})$. Since $\mathbb{R}$ is one dimensional it suffices to show that $D_A F$ is not zero for all $A \in \mathrm{SL}(n, \mathbb{R})$. Indeed

$$\begin{aligned}
D_A F(A) &= \frac{d}{dt}\bigg|_{t=0} F(A + tA) \\
&= \frac{d}{dt}\bigg|_{t=0} \det(A(1 + t)) \\
&= \frac{d}{dt}\bigg|_{t=0} (1 + t)^n \det A \\
&= \frac{d}{dt}\bigg|_{t=0} (1 + t)^n \\
&= n \neq 0.
\end{aligned}$$

So $D_A F$ is surjective for $A \in \mathrm{SL}(n, \mathbb{R})$ and $\dim(\mathrm{SL}(n, \mathbb{R})) = n^2 - 1$.

(ii) Consider the open subset $W := \{A \in \mathbb{R}^{n\times n} : \det A > 0\} \subset \mathbb{R}^{n\times n}$ and the smooth map $F : W \to \mathbb{R}^{n(n+1)/2} \cong \mathrm{Symm}(n), A \to AA^{\mathrm{T}}$. Notice that

$$\begin{aligned} F^{-1}(I) &= \left\{A \in \mathbb{R}^{n\times n} : \det A > 0, AA^{\mathrm{T}} = I\right\} \\ &= \left\{A \in \mathbb{R}^{n\times n} : \det A = 1, AA^{\mathrm{T}} = I\right\} \\ &= \mathrm{SO}(n,\mathbb{R}). \end{aligned}$$

We want to show that I is a regular value of F, that is, $D_AF : \mathbb{R}^{n\times n} \longrightarrow \mathbb{R}^{n(n+1)/2}$ is surjective for all A in $\mathrm{SO}(n,\mathbb{R})$. For $B \in \mathbb{R}^{n\times n}$ we compute

$$\begin{aligned} D_AF(B) &= \frac{d}{dt}\Big|_{t=0} F(A+tB) \\ &= \frac{d}{dt}\Big|_{t=0} (A+tB)(A+tB)^{\mathrm{T}} \\ &= \frac{d}{dt}\Big|_{t=0} (A+tB)\left(A^{\mathrm{T}}+tB^{\mathrm{T}}\right) \\ &= \frac{d}{dt}\Big|_{t=0} AA^{\mathrm{T}} + t\left(BA^{\mathrm{T}}+AB^{\mathrm{T}}\right) + t^2BB^{\mathrm{T}} \\ &= BA^{\mathrm{T}} + AB^{\mathrm{T}}. \end{aligned}$$

Hence given any X in $\mathbb{R}^{n(n+1)/2} \cong \mathrm{Symm}(n)$, set $B := \frac{1}{2}XA$, then

$$D_AF(B) = \frac{1}{2}XAA^{\mathrm{T}} + A\left(\frac{1}{2}XA\right)^{\mathrm{T}} = \frac{1}{2}XAA^{\mathrm{T}} + \frac{1}{2}AA^{\mathrm{T}}X^{\mathrm{T}} = X.$$

This shows that D_AF is surjective for all $A \in F^{-1}(I)$ and hence $\mathrm{SO}(n,\mathbb{R})$ is a $\frac{n(n-1)}{2}$-dimensional submanifold of $\mathbb{R}^{n\times n}$.

3. Tangent bundle

Let $M \subset \mathbb{R}^n$ be an m-dimensional submanifold. Show that the *tangent bundle*

$$TM := \bigcup_{p\in M} \{p\} \times TM_p$$

is a $2m$-dimensional submanifold of $\mathbb{R}^{2n}$.

Solution:

$TM \subset \mathbb{R}^{2n}$ is a $2m$-submanifold of $\mathbb{R}^{2n}$ if and only if for all $(p_0, X_0) \in TM$ there exist open sets $U \subset \mathbb{R}^{2m}, V \subset \mathbb{R}^{2n}$, an immersion $f : U \to \mathbb{R}^{2n}$ such that $(p_0, X_0) \in f(U) = TM \cap V$ and $f : U \to TM \cap V$ is a homeomorphism.

Let $(p_0, X_0) \in TM$ then there exists a local parametrization: open sets $U_0 \subset \mathbb{R}^m, V_0 \subset \mathbb{R}^n$, an immersion $\varphi : U_0 \to \mathbb{R}^n$ with $p_0 \in \varphi(U_0) = M \cap V_0$ and $\varphi : U_0 \to M \cap V_0$ is a homeomorphism.

We define a local parametrization of TM at (p_0, X_0) as follows: let $U := U_0 \times \mathbb{R}^m \subset \mathbb{R}^{2m}, V := V_0 \times \mathbb{R}^n \subset \mathbb{R}^{2n}$ and $f : U \to \mathbb{R}^{2n}$ defined by

$$f(x,\xi) = \left(\varphi(x), D_x\varphi(\xi)\right).$$

Since φ is smooth, the same holds for f. The derivative of f at (x,ξ) is given by

$$D_{(x,\xi)}f = \begin{pmatrix} D_x\varphi & 0 \\ * & D_x\varphi \end{pmatrix},$$

(using linearity of $D_x\varphi$ we have

$$D_\xi(D_x\varphi)(X) = \frac{d}{dt}\Big|_{t=0} D_x\varphi(\xi+tX) = \frac{d}{dt}\Big|_{t=0} D_x\varphi(\xi) + tD_x\varphi(X) = D_x\varphi(X).$$

) As φ is an immersion $D_x\varphi$ has rank m, so $D_{(x,\xi)}f$ has rank $2m$ and we conclude that f is an immersion.

Moreover

$$\begin{aligned} f(U) = f(U_0\times\mathbb{R}^m) &= \bigcup_{x\in U_0}\{\varphi(x)\}\times D_x\varphi(\mathbb{R}^m) = \bigcup_{p\in\varphi(U_0)}\{p\}\times TM_p \\ &= \bigcup_{p\in M\cap V_0}\{p\}\times TM_p = TM\cap(V_0\times\mathbb{R}^n) = TM\cap V, \end{aligned}$$

and $(p_0,X_0)\in f(U)$.

In order to show that $f: U\to TM\cap V$ is a homeomorphism, consider $g: TM\cap V \longrightarrow U$

$$g(p,X) := \left(\varphi^{-1}(p), D_p\varphi^{-1}(X)\right).$$

The function φ^{-1} is continuous by our previous assumption. By chain rule We also note that for $x\in U_0$, if $v\in\mathbb{R}^m$ such that $(D_x\varphi)^T D_x\varphi(v)=0$, then

$$|D_x\varphi(v)|^2 = \left\langle (D_x\varphi)^T D_x\varphi(v), v\right\rangle = 0,$$

hence $v=0$ since $D_x\varphi$ is injective. It follows that $(D_x\varphi)^T D_x\varphi$ is non-singular and thus $(D_x\varphi)^T$ is injective on $\mathrm{Range}(D_x\varphi)$. Then for $X\in\mathrm{Range}(D_x\varphi)$, we have

$$D_x\varphi(\xi)=X \Leftrightarrow (D_x\varphi)^T D_x\varphi(\xi) = (D_x\varphi)^T X \Leftrightarrow \xi = \left((D_x\varphi)^T D_x\varphi\right)^{-1}(D_x\varphi)^T X.$$

So by chain rule we have $D_p\varphi^{-1}(X) = \left((D_{\varphi^{-1}(p)}\varphi)^T D_{\varphi^{-1}(p)}\varphi\right)^{-1}(D_{\varphi^{-1}(p)}\varphi)^T X$ is continuous with respect to $(p,X)\in TM\cap V$.

Finally, we have

$$\begin{aligned} f\circ g(p,X) &= f\left(\varphi^{-1}(p), D_p\varphi^{-1}(X)\right) \\ &= \left(p, D_{\varphi^{-1}(p)}\varphi\, D_p\varphi^{-1}(X)\right) \\ &= \left(p, D_p(\varphi\circ\varphi^{-1})(X)\right) \\ &= (p,X), \end{aligned}$$

and similarly

$$g\circ f(x,\xi) = g\left(\varphi(x), D_x\varphi(\xi)\right) = \left(\varphi^{-1}\circ\varphi(x), D_{\varphi(x)}\varphi^{-1}D_x\varphi(\xi)\right) = (x,\xi).$$

This shows that $f^{-1}=g$ is continuous.

(Remark: for Hausdorff spaces X,Y and an injective continuous map $f: X\to f(X)\subset Y$, we have a general condition on f to guarantee it is a homeomorphism onto $f(X)$:

$$\forall y\in f(X), \exists \text{ an open neighborhood } U\subset Y \text{ of } y \text{ and a compact set } K\subset X \text{ such that } f^{-1}(U)\subset K. \quad (\star)$$

Suppose $(\star)$ holds, it suffices to show that $f(C)$ is closed in $f(X)$ for any closed $C\subset X$. Let $y\in f(X)\backslash f(C)$, we choose U and K as in $(\star)$. Then $K\cap C$ is a closed subset of a compact set, hence compact. We have

$f(K \cap C)$ is compact since f is continuous. Note that $y \notin f(C)$, so in particular, $y \notin f(K \cap C)$. Since Y is Hausdorff, there exists an open neighborhood V of y such that $V \cap f(K \cap C) = \emptyset$. Therefore, $U \cap V$ is a neighborhood of y such that $U \cap V \cap f(C) = \emptyset$ (suppose $y' \in U \cap V$ and $y' = f(x)$ for $x \in C$, then $x \in K$ since $f^{-1}(U) \subset K$. Hence $y' \in f(K \cap C) \cap V$, contradiction arises). Since y is arbitrarily chosen in $f(X) \setminus f(C)$, we have $f(C)$ is closed.

The advantage of this approach is to avoid explicitly writing the inverse of f and in many cases we only need some "rough" estimates. As an application, to show f is a homeomorphism in the above solution, we firstly note that $f : U \to TM \cap V$ is bijective since $\varphi : U_0 \to M \cap V_0$ is a homeomorphism and $D_x\varphi$ is injective for $x \in U_0$. Let $(p, X) \in TM \cap V$. Since $\varphi : U_0 \to M \cap V_0$ is a homeomorphism there exists $\varepsilon > 0$ and a compact set $K \subset U_0$ such that $B_\varepsilon(p) \cap M \subset \varphi(K)$. Since K is compact, we have $\big((D_x\varphi)^T D_x\varphi\big)^{-1}$ is continuous hence bounded in the set of $m \times m$ positive definite matrices for $x \in K$. Thus we have a positive lower bound for the smallest eigenvalue of $(D_x\varphi)^T D_x\varphi$ when $x \in K$. As a result, there exists $C > 0$ such that

$$|D_x\varphi(v)|^2 = \big\langle (D_x\varphi)^T D_x\varphi(v), v\big\rangle \geq C^2|v|^2$$

whenever $x \in K$, $v \in \mathbb{R}^m$. Therefore, if $f(x, \xi) \in B_\varepsilon\big((p, X)\big)$, then $x \in K$ and

$$|\xi| \leq C^{-1}|D_x\varphi(\xi)| \leq C^{-1}(|X| + \varepsilon),$$

i.e., $f^{-1}\Big(B_\varepsilon\big((p, X)\big)\Big) \subset K \times \overline{B_{C^{-1}(|X|+\varepsilon)}(0)} \subset U$. So the condition $(\star)$ is satisfied and f is a homeomorphism.)