# Solution 3

**1. Differentiability**

Let $M \subset \mathbb{R}^n$ be an $m$-dimensional submanifold, $F : M \longrightarrow \mathbb{R}^k$ a map and $p \in M$. Show that the following statements are equivalent.

(i) $F$ is differentiable in $p$ (as defined in the lecture).

(ii) There exists an open neighborhood $V$ of $p$ in $\mathbb{R}^n$ and a map $\bar{F} : V \longrightarrow \mathbb{R}^k$ differentiable at $p$ with

$$\bar{F}|_{V \cap M} = F|_{V \cap M}.$$

## Solution:

$(i) \Rightarrow (ii)$. By definition if $F$ is differentiable at $p$, then there exists an open neighborhood $U \subset \mathbb{R}^m$ of 0 and a local parametrization $f : U \longrightarrow f(U) \subset M$ with $f(0) = p$ such that $F \circ f : U \longrightarrow \mathbb{R}^k$ is differentiable at $0 \in U$.

By the implicit function theorem (injective version, A.3 in the lecture notes) applied to $f$, there exist open neighborhoods $V \subset \mathbb{R}^n$ of $p$, $W \subset U \times \mathbb{R}^{n-m}$ of $(0, 0)$ and a diffeomorphism $\varphi : V \longrightarrow W$ such that $\varphi(p) = (0, 0)$ and $(\varphi \circ f)(x) = (x, 0)$ for all $(x, 0) \in W$.

Denote by $\pi : W \longrightarrow U$ the projection onto the first $m$-coordinates and set $U' := \{x \in U : (x, 0) \in W\}$. Notice that $U'$ is an open neighborhood of 0 and $(\varphi \circ f)(x) = (x, 0)$ for all $x \in U'$. Now for any $q \in f(U')$, there exists a neighborhood $B_q \subset V$ of $q$ such that $B_q \cap M \subset f(U')$ since $f$ is a local parametrization. Let $V' := \bigcup_{q \in f(U')} B_q \subset V$. Then we have $p \in V'$ and $V' \cap M \subset f(U')$. Define $\bar{F} : V' \longrightarrow \mathbb{R}^k$ by

$$\bar{F} := (F \circ f) \circ \pi \circ \varphi|_{V'}.$$

The map $\bar{F}$ is differentiable at $p$ and it agrees with $F$ on $V' \cap M$, indeed for $q = f(y) \in V' \cap M \subset f(U')$:

$$\bar{F}(q) = F \circ f \circ \pi \circ \varphi \circ f(y) = F \circ f \circ \pi(y, 0) = F \circ f(y) = F(q)$$

Remark: We needed to introduce the sets $U'$, $V'$ because there might be points $q \in V \cap M$ which are not in $f(U)$ (this is similar to one of the proofs seen in class).

$(ii) \Rightarrow (i)$. Let $V \subset \mathbb{R}^n$ be an open neighborhood of $p$ and let $\bar{F} : V \longrightarrow \mathbb{R}^k$ be a map which is differentiable at $p$ and such that $\bar{F}|_{V \cap M} = F|_{V \cap M}$.

Let $f : U \longrightarrow \mathbb{R}^n$ be a local parametrization of $M$ at $p$ with $f(x) = p$, $x \in U$ and $f(U) \subset V \cap M$. Then $F \circ f = \bar{F} \circ f$ is differentiable at $p$ and so by definition $F$ is differentiable at $p$.

**2. Orientability**

(i) Let $W \subset \mathbb{R}^n$ be an open set, $f : W \longrightarrow \mathbb{R}$ a $C^1$-map and $y \in \mathbb{R}$ a regular value of $f$. Prove that $M := f^{-1}(\{y\})$ is an orientable submanifold.

(ii) Prove that the submanifolds $\mathrm{SL}(n, \mathbb{R})$ and $\mathrm{SO}(n, \mathbb{R})$ are orientable.

## Solution:

(i) By the regular value theorem $M$ is a $(n-1)$-submanifold of $\mathbb{R}^n$ so in order to show that it's orientable we'll construct a Gauss map $N : M \longrightarrow S^{n-1}$ (see Proposition 2.10 in the notes).

Since $y$ is a regular value, it holds that $df_p$ has rank 1 for all $p$ in $M$ and in particular $\nabla f(p) \neq 0$. We define

$$N : M \longrightarrow S^{n-1}, \quad p \longmapsto \frac{\nabla f(p)}{|\nabla f(p)|}.$$

As the gradient, if not zero, is always perpendicular to the level sets of $f$ (let $\gamma : (-\varepsilon, \varepsilon) \to M$ with $\gamma(0) = p$, then $f(\gamma(t)) \equiv y \Rightarrow 0 = \frac{d}{dt}|_{t=0} f(\gamma(t)) = df_p(\gamma'(0)) = \langle \nabla f(p), \gamma'(0) \rangle$), this defines the desired Gauss map.

(ii) We showed in Exercise Sheet 2 that $\mathrm{SL}(n, \mathbb{R}) = \det^{-1}(1)$ is the preimage of a regular value of a map $\mathbb{R}^{n \times n} \longrightarrow \mathbb{R}$, hence orientability follows from part (i).

For $\mathrm{SO}(n, \mathbb{R})$ we will produce a compatible system of local parametrizations. The proof uses merely the fact that $G := \mathrm{SO}(n, \mathbb{R})$ is not only a (sub)manifold, but also group, with smooth group operations $(g, h) \mapsto gh$ and $g \mapsto g^{-1}$ (that is, $G$ is a Lie group, and any Lie group is orientable).

Note first that for any $g \in G$ the left multiplication

$$L_g : G \longrightarrow G, \quad L_g(A) = gA$$

is a smooth map: clearly the (linear) map $A \mapsto gA$ from $\mathbb{R}^{n \times n}$ to $\mathbb{R}^{n \times n}$ is smooth, and thus the restriction to $G$ is smooth (compare Exercise 1). Similarly, $L_{g^{-1}}$ is smooth, and $L_{g^{-1}} \circ L_g = \mathbb{1}_G$, so $L_g$ is in fact a diffeomorphism of $G$.

Now pick a basis $(X_1, \dots, X_m)$ of the tangent space $TG_e$ at the neutral element, and put $X_i(g) := d(L_g)_e(X_i)$ for every $g \in G$ and $i = 1, \dots, m$ (in fact, since $L_g$ is the restriction of a linear map, $X_i(g)$ is just $gX_i$, but this is irrelevant). Since $L_g$ is a diffeomorphism of $G, (X_1(g), \dots, X_m(g))$ is a basis of $TG_g$ for every $g \in G$. Furthermore, every $X_i$ is a smooth vector field on $G$ by smoothness of the group operations.

For every $g \in G$ there exists a local parametrization

$$f_g : U_g \longrightarrow f_g(U_g) \subset G$$

such that $U_g$ is a connected open neighborhood of $0 \in \mathbb{R}^m, f_g(0) = g$, and $d(f_g)_0(e_i) = X_i(g)$ for $i = 1, \dots, m$, where $\{e_i\}$ is the standard basis of $\mathbb{R}^m$. To show that $\{f_g\}_{g \in G}$ is a compatible system, suppose that $g, h \in G$ and $x \in U_g, y \in U_h$ are such that $f_g(x) = f_h(y) =: p$. Then the basis $(d(f_g)_{x'}(e_1), \dots, d(f_g)_{x'}(e_m))$ of $TG_{f_g(x')}$ is equivalent to $(X_1(f_g(x')), \dots, X_m(f_g(x')))$ for any $x' \in U_g$ (since $U_g$ is connected and the determinant is non-zero and continuous on $U_g$, the sign of determinant must be constantly positive). In particular, the basis $(d(f_g)_x(e_1), \dots, d(f_g)_x(e_m))$ of $TG_p$ is equivalent to $(X_1(p), \dots, X_m(p))$. Likewise, $(d(f_h)_y(e_1), \dots, d(f_h)_y(e_m))$ is equivalent to $(X_1(p), \dots, X_m(p))$, and so

$$(d(f_g)_x(e_1), \dots, d(f_g)_x(e_m)) \quad \text{and} \quad (d(f_h)_y(e_1), \dots, d(f_h)_y(e_m))$$

are equivalent. Put $V := f_g(U_g) \cap f_h(U_h)$ and $\psi := f_h^{-1} \circ f_g : f_g^{-1}(V) \longrightarrow f_h^{-1}(V)$. Since $d(f_g)_x = d(f_h)_y \circ d\psi_x$, we have $\det(d\psi_x) > 0$.

**3. Angle-preserving parametrization**

Let $U \subset \mathbb{R}^m$ be an open set and $f : U \longrightarrow \mathbb{R}^n$ an immersion. The map $f$ is called *angle-preserving* if for all $x \in U$ and $\xi, \eta \in TU_x$ the angles between $\xi, \eta$ and $df_x(\xi), df_x(\eta)$ coincide. Here the angles are meant with respect to the standard scalar products on $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively.

(i) Show that $f$ is angle-preserving if and only if $g_{ij} = \lambda^2 \delta_{ij}$, where $g_{ij}$ is the matrix of the first fundamental form of $f$, $\delta_{ij}$ is the Kronecker delta and $\lambda : U \longrightarrow \mathbb{R}$ is a differentiable function.

(ii) Find an angle-preserving parametrization of the 2 -sphere without North and South Pole, $S^2 \backslash \{N, S\} \subset \mathbb{R}^3$, of the form

$$f(x,y) = \left(\sqrt{1-h^2(y)}\cos(x), \sqrt{1-h^2(y)}\sin(x), h(y)\right),$$

where $h : \mathbb{R} \longrightarrow \mathbb{R}$ is a differentiable odd function.

## Solution:

(i) Notice first that for an immersion, the condition of being angle-preserving corresponds to: for all $x \in U$ and $\xi, \eta \in \mathbb{R}^m \setminus \{0\}$,

$$\frac{\langle df_x(\xi), df_x(\eta)\rangle_n}{|df_x(\xi)|_n |df_x(\eta)|_n} = \frac{\langle \xi, \eta\rangle_m}{|\xi|_m |\eta|_m},$$

where $\langle \cdot, \cdot \rangle_m$ and $\langle \cdot, \cdot \rangle_n$ denote the Euclidean scalar products in $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively (these indices will be omitted in the following).

Suppose that $f$ is angle preserving. Then the matrix of the first fundamental form $g$ of $f$ in $x$ satisfies

$$\begin{aligned} g_{ij}(x) &= g_x(e_i, e_j) = \langle df_x(e_i), df_x(e_j)\rangle \\ &= |df_x(e_i)|\,|df_x(e_j)|\langle e_i, e_j\rangle = |df_x(e_i)|\,|df_x(e_j)|\,\delta_{ij}. \end{aligned}$$

We now claim that $|df_x(e_i)| = |df_x(e_j)|$ for all $x \in U$ and $i, j$. Indeed,

$$\begin{aligned} |df_x(e_i)|^2 - |df_x(e_j)|^2 &= \langle df_x(e_i), df_x(e_i)\rangle - \langle df_x(e_j), df_x(e_j)\rangle \\ &= \langle df_x(e_i) - df_x(e_j), df_x(e_i) + df_x(e_j)\rangle \\ &= \langle df_x(e_i - e_j), df_x(e_i + e_j)\rangle \\ &= \frac{|df_x(e_i - e_j)| \cdot |df_x(e_i + e_j)|}{|e_i - e_j| \cdot |e_i + e_j|}\langle e_i - e_j, e_i + e_j\rangle = 0. \end{aligned}$$

Thus $g_{ij} = \lambda^2 \delta_{ij}$ for $\lambda : U \longrightarrow \mathbb{R}, x \mapsto |df_x(e_1)|$, which is differentiable since $f$ is an immersion and hence $df_x(e_1) \neq 0$ for any $x \in U$.

Suppose now that $g_{ij} = \lambda^2 \delta_{ij}$, then $\lambda \neq 0$ on $U$ since $f$ is an immersion, and for $\xi = (x_1, \dots, x_m), \eta = (y_1, \dots, y_m) \in \mathbb{R}^m$ we have

$$\langle df_x(\xi), df_x(\eta)\rangle = \sum_{i,j=1}^m x_i y_j \langle df_x(e_i), df_x(e_j)\rangle = \sum_{i,j=1}^m x_i y_j g_{ij} = \lambda^2 \sum_{i=1}^m x_i y_i = \lambda^2 \langle \xi, \eta\rangle.$$

In particular, this implies $|df_x(\xi)| = |\lambda|\,|\xi|$ for any $\xi \in \mathbb{R}^m$. Consequently,

$$\frac{\langle df_x(\xi), df_x(\eta)\rangle}{|df_x(\xi)|\,|df_x(\eta)|} = \frac{\lambda^2\langle \xi, \eta\rangle}{\lambda^2|\xi|\,|\eta|} = \frac{\langle \xi, \eta\rangle}{|\xi|\,|\eta|}$$

for any $\xi, \eta \in \mathbb{R}^m \setminus \{0\}$, so $f$ is angle-preserving.

(ii) We compute

$$J_f = \begin{pmatrix} -\sqrt{1-h(y)^2}\sin x & -\frac{h(y)h'(y)}{\sqrt{1-h(y)^2}}\cos x \\ \sqrt{1-h(y)^2}\cos x & -\frac{h(y)h'(y)}{\sqrt{1-h(y)^2}}\sin x \\ 0 & h'(y) \end{pmatrix}$$

and

$$(g_{ij}) = J_f^T J_f = \begin{pmatrix} 1-h(y)^2 & 0 \\ 0 & \frac{h'(y)^2}{1-h(y)^2} \end{pmatrix}.$$

From part (i) we know that $f$ is angle-preserving if

$$1-h(y)^2 = \frac{h'(y)^2}{1-h(y)^2}. \tag{1}$$

Since $f$ is a parametrization of $S^2 \setminus \{N,S\}$, $h \in (-1,1)$ and we may assume without loss of generality that $h' \geq 0$. Then the differential equation (1) reduces to

$$1-h(y)^2 = h'(y). \tag{2}$$

Let $g(x) := \frac{1}{2}\log\frac{1+x}{1-x}$ for $x \in (-1,1)$. If $h$ satisfies equation (2), then

$$(g \circ h)'(y) = g'(h(y))\,h'(y) = \frac{h'(y)}{1-h(y)^2} = 1,$$

so $g(h(y)) = y + g(h(0)) = y$ since $h$ is odd. Therefore, we obtain the solution

$$h(y) = \frac{e^{2y}-1}{e^{2y}+1} = \tanh(y)$$

for $y \in \mathbb{R}$ (Another solution is of course $h(y) = -\tanh(y)$).