Solution 4

1. Tubular surface

Let $c : [0, L] \to \mathbb{R}^3$ be a smooth Frenet curve, parametrized by arc-length with normal vector n and binormal vector b. Show that if $r > 0$ is sufficiently small, then the tubular surface $f : [0, L] \times \mathbb{R} \to \mathbb{R}^3$ around c defined by

$$f(t, \varphi) := c(t) + r\big(\cos\varphi \cdot n(t) + \sin\varphi \cdot b(t)\big)$$

is regular and the area of $f|_{[0,L]\times[0,2\pi)}$ equals $2\pi r L$.

Solution:

Recall that $e := e_1 = \dot{c}$, $n := e_2$ and $b := e_3 = e \times n$. Hence we have the relations

$$e \times n = b, \quad b \times e = n, \quad n \times b = e.$$

With this notation we also have

$$\begin{pmatrix} \dot{e} \\ \dot{n} \\ \dot{b} \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} e \\ n \\ b \end{pmatrix}.$$

In order to show that f is an immersion for r small enough, we begin by computing $\frac{\partial f}{\partial t}(t,\varphi)$ and $\frac{\partial f}{\partial \varphi}(t,\varphi)$:

$$\begin{aligned}
\frac{\partial f}{\partial t}(t,\varphi) &= \dot{c}(t) + r\big(\cos\varphi \cdot \dot{n}(t) + \sin\varphi \cdot \dot{b}(t)\big) \\
&= \dot{c}(t) + r\Big(\cos\varphi\big(-\kappa(t)\cdot e(t) + \tau(t)\cdot b(t)\big) - \tau(t)\sin\varphi \cdot n(t)\Big) \\
&= \big(1 - r\kappa(t)\cos\varphi\big)e(t) + r\tau(t)\cos\varphi \cdot b(t) - r\tau(t)\sin\varphi \cdot n(t),
\end{aligned}$$

and

$$\frac{\partial f}{\partial \varphi}(t,\varphi) = -r\sin\varphi \cdot n(t) + r\cos\varphi \cdot b(t).$$

Using the above relations, a computation shows that

$$\frac{\partial f}{\partial t}(t,\varphi) \times \frac{\partial f}{\partial \varphi}(t,\varphi) = r\big(r\kappa(t)\cos\varphi - 1\big)\big(\sin\varphi \cdot b(t) + \cos\varphi \cdot n(t)\big),$$

hence

$$\Big|\frac{\partial f}{\partial t}(t,\varphi) \times \frac{\partial f}{\partial \varphi}(t,\varphi)\Big| = |r(r\kappa(t)\cos\varphi - 1)|.$$

This shows that if $r < \min_{t\in[0,L]} \frac{1}{\kappa(t)}$, then $1 - r\kappa(t)\cos\varphi > 0$, so $|\frac{\partial f}{\partial t}(t,\varphi) \times \frac{\partial f}{\partial \varphi}(t,\varphi)| \neq 0$ and $\frac{\partial f}{\partial t}(t,\varphi)$, $\frac{\partial f}{\partial \varphi}(t,\varphi)$ are linearly independent. Therefore $df_{t,\varphi}$ has rank 2 and f is an immersion.

In order to computer the area of $f|_{[0,L]\times[0,2\pi)}$, we need to compute the determinant of the matrix (g_{ij}) of the first fundamental form of f:

$$\sqrt{\det(g_{ij}(t,\varphi))} = \text{Area}(\text{span}(\frac{\partial f}{\partial t}(t,\varphi), \frac{\partial f}{\partial \varphi}(t,\varphi))) = \big|\frac{\partial f}{\partial t}(t,\varphi) \times \frac{\partial f}{\partial \varphi}(t,\varphi)\big|.$$

Thus

$$\begin{aligned}\text{Area}(f|_{[0,L]\times[0,2\pi)}) &= \int_0^L \int_0^{2\pi} r(1 - r\kappa(t)\cos\varphi)\, d\varphi dt \\ &= \int_0^L (2\pi r - 0)\, dt \\ &= 2\pi r L.\end{aligned}$$

2. Torus

Let $a > r > 0$ and $f : \mathbb{R}^2 \to \mathbb{R}^3$ be the parametrization of a torus T, defined as

$$f(x, y) := \big((a + r\cos x)\cos y, (a + r\cos x)\sin y, r\sin x\big).$$

Prove that:

(a) If a geodesic is at some point tangential to the circle $x = \frac{\pi}{2}$ then it must be contained in the region of T with $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$.

(b) Suppose that a geodesic $c\colon \mathbb{R} \to T$, which crosses the circle $x = 0$ with an angle $\theta \in (0, \frac{\pi}{2})$, also intersects the circle $x = \pi$, then

$$\cos\theta \le \frac{a - r}{a + r}.$$

Solution:

(a) Suppose that the geodesic $c\colon I \to T$ intersects the circle $x = \pi/2$ tangentially at time t_0. We write $\theta(t) \in [0, \pi]$ for the angle between $\dot{c}(t)$ and the horizontal circle intersecting $c(t)$ at time t and $r(t) = a + r\cos(x(t))$ for the distance between $c(t)$ and the z-axis.

It is given that $r(t_0) = a$ and $\theta(t_0) = 0$. By Clairaut's condition

$$r(t)\cos\theta(t) = r(t_0)\cos\theta(t_0) = a$$

for all $t \in I$ and hence $r(t) \ge a$ for all $t \in I$. We conclude that $-\pi/2 \le x(t) \le \pi/2$.

(b) Suppose that the geodesic c intersects the horizontal circle $x = 0$ at time t_0 with an angle $\theta(t_0) = \theta$ and the horizontal circle $x = \pi$ at time t_1 with an angle $\theta(t_1)$. By Clairaut's condition it holds that

$$a - r \ge r(t_1)\cos\theta(t_1) = r(t_0)\cos\theta(t_0) = (a + r)\cos\theta,$$

from which we deduce that

$$\cos\theta \le \frac{a - r}{a + r}.$$

3. Energy

Let $M \subset \mathbb{R}^n$ be a submanifold, $c_0 : [a, b] \to M$ a smooth curve and

$$E(c_0) := \frac{1}{2}\int_a^b |\dot{c}_0(t)|^2\, dt$$

its energy.

(a) Show that $L(c_0)^2 \leq 2(b-a)E(c_0)$ with an equality if and only if c_0 is parametrized proportionally to arc-length.

(b) Compute $\frac{d}{ds}\big|_{s=0} E(c_s)$ for a smooth variation $\{c_s\}_{s\in(-\varepsilon,\varepsilon)}$ of c_0 in M.

(c) Characterize geodesics in M using the energy.

Solution:

(a) From the Cauchy-Schwarz inequality it follows that

$$\begin{aligned} L(c_0)^2 &= \left(\int_a^b |\dot c_0(t)|\,dt\right)^2 = \left(\int_a^b |\dot c_0(t)|\cdot 1\,dt\right)^2 \\ &\leq \int_a^b |\dot c_0|^2\,dt \cdot \int_a^b 1^2\,dt = 2(b-a)E(c_0). \end{aligned}$$

Equality holds if and only if $|\dot c_0|$ and 1 are linearly dependent in $L^2([a,b])$, that is, if and only if $|\dot c_0|$ is constant.

(b) Denote by $V_s(t) := \frac{\partial}{\partial s}c_s(t)$ the variation vector field associated to the variation $\{c_s\}$. Then we claim that

$$\frac{d}{ds}\bigg|_{s=0} E(c_s) = g(\dot c_0(t), V_0(t))\Big|_a^b - \int_a^b g\left(\frac{D}{dt}\dot c_0(t), V_0(t)\right) dt.$$

Indeed,

$$\begin{aligned} \frac{d}{ds}\bigg|_{s=0} E(c_s) &= \frac{d}{ds}\bigg|_{s=0} \frac{1}{2}\int_a^b g(\dot c_s(t), \dot c_s(t))\,dt \\ &= \frac{1}{2}\int_a^b \frac{d}{ds}\bigg|_{s=0} g(\dot c_s(t), \dot c_s(t))\,dt \\ &= \int_a^b g\Big(\dot c_0(t), \frac{d^2}{ds\,dt}\bigg|_{s=0} c_s(t)\Big)\,dt \\ &= \int_a^b g\Big(\dot c_0(t), \frac{d}{dt}V_0(t)\Big)\,dt \\ &= \int_a^b g\Big(\dot c_0(t), \frac{D}{dt}V_0(t)\Big)\,dt \\ &= \int_a^b \frac{d}{dt} g(\dot c_0(t), V_0(t))\,dt - \int_a^b g\Big(\frac{D}{dt}\dot c_0(t), V_0(t)\Big)\,dt \\ &= g(\dot c_0(t), V_0(t))\Big|_a^b - \int_a^b g\Big(\frac{D}{dt}\dot c_0(t), V_0(t)\Big)\,dt. \end{aligned}$$

(c) We claim that $c_0 : [a,b] \to M$ is a geodesic if and only if $\frac{d}{ds}\big|_{s=0} E(c_s) = 0$ for all proper variations $\{c_s\}_{s\in(-\varepsilon,\varepsilon)}$ of c_0 in M.

First notice that $V_0(a) = V_0(b) = 0$ for the variation vector field V_s of a proper variation, hence from (b):

$$\frac{d}{ds}\bigg|_{s=0} E(c_s) = -\int_a^b g\left(\frac{D}{dt}\dot c_0(t), V_0(t)\right) dt.$$

If c_0 is a geodesic, then $\frac{D}{dt}\dot{c}_0(t) = 0$ and hence

$$\frac{d}{ds}\bigg|_{s=0} E(c_s) = 0.$$

On the other hand if c_0 is not a geodesic, there exists $t_0 \in (a,b)$ with $\frac{D}{dt}\dot{c}_0(t_0) \neq 0$. Let $f\colon U \subset \mathbb{R}^m \to M$ be a local parametrization of M with $f(0) = c(t_0)$. Set $\xi := (df_0)^{-1}\left(\frac{D}{dt}\dot{c}_0(t_0)\right)$. Take $r, \delta > 0$ small enough such that $[t_0 - r, t_0 + r] \subset [a,b]$, $c(t) \in f(U)$ for all $t \in [t_0 - r, t_0 + r]$ and

$$\left\langle \frac{D}{dt}\dot{c}_0(t), df_{\gamma_0(t)}(\xi) \right\rangle \geq \delta$$

for all $t \in [t_0 - r, t_0 + r]$, where $\gamma_0 := f^{-1} \circ c_0$. (This is possible since $df_{\gamma_0(t_0)} = df_{f^{-1}(c_0(t_0))} = df_0$.)

Take $h\colon [a,b] \to [0,1]$ a smooth function with

$$h(t) = \begin{cases} 1, & |t - t_0| \leq \frac{r}{2}, \\ 0, & |t - t_0| \geq \frac{3r}{4}. \end{cases}$$

Now we'll define a proper variation of $c = c_0$. Since $\gamma_0([t_0 - r, t_0 + r])$ is a compact subset of U, there exists $\varepsilon > 0$ such that $\gamma_0(t) + v \in U$ for any $t \in [t_0 - r, t_0 + r]$, $v \in \mathbb{R}^m$ with $|v| \leq \varepsilon|\xi|$. For $s \in (-\varepsilon, \varepsilon)$, let $c_s : [a,b] \to M$

$$c_s(t) := \begin{cases} c_0(t), & |t - t_0| \geq r, \\ f\left(\gamma_0(t) - s\, h(t) \cdot \xi\right), & |t - t_0| \leq r. \end{cases}$$

Then

$$V_0(t) = \frac{d}{ds}\bigg|_{s=0} c_s(t) = \begin{cases} 0, & |t - t_0| \geq r, \\ df_{\gamma_0(t)}(-h(t) \cdot \xi), & |t - t_0| \leq r, \end{cases}$$

hence

$$\begin{aligned} \frac{d}{ds}\bigg|_{s=0} E(c_s) &= -\int_a^b g\left(\frac{D}{dt}\dot{c}_0(t), V_0(t)\right) dt \\ &= -\int_{t_0-r}^{t_0+r} \left\langle \frac{D}{dt}\dot{c}_0(t), df_{\gamma_0(t)}(-h(t) \cdot \xi) \right\rangle dt \\ &= \int_{t_0-r}^{t_0+r} h(t) \cdot \left\langle \frac{D}{dt}\dot{c}_0(t), df_{\gamma_0(t)}(\xi) \right\rangle dt \\ &\geq \int_{t_0-\frac{r}{2}}^{t_0+\frac{r}{2}} \delta\, dt \\ &= r\delta > 0. \end{aligned}$$

Therefore we have found a proper variation of c with $\frac{d}{ds}\big|_{s=0} E(c_s) \neq 0$.