Solution 5

1. Elliptic Points

A point $p \in M \subset \mathbb{R}^{m+1}$ on a hypersurface is called *elliptic* if the second fundamental form is (positive or negative) definite. Show that if M is compact then it has elliptic points.

Solution:

Since M is compact, it is closed and bounded. Hence there exists a radius $R > 0$ such that M is contained in $\bar{B}_R(0)$ and the boundary $S := S^m_R(0) = \partial \bar{B}_R(0)$ touches M in (at least) one point $p \in M$.

As S touches M in p, it holds that $TS_p = TM_p$. In a neighborhood of p one can write M as a graph over TM_p: let f be such a local parametrization, so (upon a translation and an orthogonal transformation if necessary) we have

$$f(x^1, \ldots, x^m) = (x^1, \ldots, x^m, b(x^1, \ldots, x^m))$$

with $b(0, \ldots, 0) = 0$ and $\nabla b(0, \ldots, 0) = 0$.

As seen in class the matrix of the second fundamental form of f is given by

$$(h_{ij}) = \frac{1}{\sqrt{1 + |\nabla b|^2}} \mathrm{Hess}(b),$$

where $\mathrm{Hess}(b) := (b_{ij})$ is the Hessian matrix of b. In particular it holds that $(h_{ij}(0)) = \mathrm{Hess}_0(b) = (b_{ij}(0))$.

The sphere S can also be locally parametrized around p by

$$g(x^1, \ldots, x^m) := (x^1, \ldots, x^m, s(x^1, \ldots, x^m))$$

with $s(x) = R - \sqrt{R^2 - |x|^2}$. Notice that $s(0) = 0$, $\nabla s = 0$ and $\mathrm{Hess}_0(s) = (s_{ij}(0)) = \frac{1}{R}\mathbb{1}$.

Moreover, since M is contained the closed ball bounded by S, we have that $b(x) \geq s(x)$. Therefore a Taylor-expansion around 0 show that

$$b(x) = \frac{1}{2} x^{\mathrm{T}} \mathrm{Hess}_0(b) x + \mathcal{O}\left(|x|^3\right) \geq s(x) = \frac{1}{2} x^{\mathrm{T}} \mathrm{Hess}_0(s) x + \mathcal{O}\left(|x|^3\right) = \frac{1}{2R}|x|^2 + \mathcal{O}\left(|x|^3\right),$$

from which we deduce that $y^{\mathrm{T}} \mathrm{Hess}_0(b) y \geq \frac{1}{R}|y|^2$ for any $y \in \mathbb{R}^m$, which shows that $(h_{ij}(0)) = \mathrm{Hess}_0(b)$ is positive definite and p is an elliptic point.

2. Mean Curvature

Let $M \subset \mathbb{R}^3$ be a surface and $p \in M$ a point. Fix $0 \neq v_0 \in TM_p$. Let $H(p)$ be the mean curvature in p and denote by $\kappa_p(\theta) := h_p(v, v)$ the normal curvature in direction v, where $v \in TM_p$, $|v| = 1$, forms an angle θ with v_0.

Prove that

$$H(p) = \frac{1}{\pi} \int_0^\pi \kappa_p(\theta)\, d\theta.$$

Solution:

Let (e_1, e_2) be an orthonormal basis of TM_p consisting of principal curvature directions, i.e. $L_p e_i = \kappa_i e_i$, for $i = 1, 2$.

If $v_0 = \lambda(\cos\theta_0 \cdot e_1 + \sin\theta_0 \cdot e_2)$ for some $\lambda > 0$, then the vector v at an angle θ with v_0 is given by

$$v(\theta) = \cos(\theta_0 + \theta) \cdot e_1 + \sin(\theta_0 + \theta) \cdot e_2.$$

Then we can compute the normal curvature as follows:

$$\begin{aligned}\kappa_p(\theta) &= h_p(v(\theta), v(\theta)) = g_p(v(\theta), L_p(v(\theta)))\\ &= \langle \cos(\theta_0 + \theta) \cdot e_1 + \sin(\theta_0 + \theta) \cdot e_2, \kappa_1 \cos(\theta_0 + \theta) \cdot e_1 + \kappa_2 \sin(\theta_0 + \theta) \cdot e_2 \rangle\\ &= \kappa_1 \cos^2(\theta_0 + \theta) + \kappa_2 \sin^2(\theta_0 + \theta)\end{aligned}$$

from which we obtain

$$\begin{aligned}\int_0^\pi k_p(\theta)\, d\theta &= \kappa_1 \cdot \int_0^\pi \cos^2(\theta_0 + \theta)\, d\theta + \kappa_2 \cdot \int_0^\pi \sin^2(\theta_0 + \theta)\, d\theta\\ &= \kappa_1 \cdot \frac{\pi}{2} + \kappa_2 \cdot \frac{\pi}{2} = \frac{1}{2}(\kappa_1 + \kappa_2) \cdot \pi = H(p) \cdot \pi,\end{aligned}$$

so $H(p) = \frac{1}{\pi} \int_0^\pi k_p(\theta)\, d\theta$.

3. Local Isometries

Let $f, \tilde{f} \colon \mathbb{R}_{\geq 0} \times \mathbb{R} \to \mathbb{R}^3$ be two immersions, given by

$$\begin{aligned}f(x, y) &:= (x \sin y, x \cos y, \log x),\\ \tilde{f}(x, y) &:= (x \sin y, x \cos y, y).\end{aligned}$$

a) Show that f and $\tilde{f}$ have the same Gauss curvature (as functions of (x, y)).

b) Are f and $\tilde{f}$ (locally) isometric?

Hint: Consider the level sets of the Gauss curvature and the curves orthogonal to these.

Solution:

a) We begin by computing the partial derivatives of f and the Gauss map:

$$\begin{aligned}f_x(x, y) &= (\sin y, \cos y, \frac{1}{x}), \quad f_y(x, y) = (x \cos y, -x \sin y, 0),\\ f_{xx}(x, y) &= (0, 0, -\frac{1}{x^2}), \quad f_{yy}(x, y) = (-x \sin y, -x \cos y, 0),\\ f_{xy}(x, y) &= f_{yx}(x, y) = (\cos y, -\sin y, 0),\\ \nu &= \frac{f_x \times f_y}{|f_x \times f_y|} = \frac{1}{\sqrt{1 + x^2}}(\sin y, \cos y, -x).\end{aligned}$$

Thus

$$(g_{ij}) = (\langle f_i, f_j \rangle) = \begin{pmatrix} 1 + \frac{1}{x^2} & 0 \\ 0 & x^2 \end{pmatrix},$$

$$(h_{ij}) = (\langle f_{ij}, \nu \rangle) = \frac{1}{\sqrt{1+x^2}} \begin{pmatrix} \frac{1}{x} & 0 \\ 0 & -x \end{pmatrix}$$

and therefore

$$K(x,y) = \frac{\det(h_{ij})}{\det(g_{ij})} = \frac{-\frac{1}{1+x^2}}{1+x^2} = -\frac{1}{(1+x^2)^2}.$$

Analogously for $\tilde{f}$ we have

$$\begin{aligned} \tilde{f}_x(x,y) &= (\sin y, \cos y, 0), \quad \tilde{f}_y(x,y) = (x\cos y, -x \sin y, 1), \\ \tilde{f}_{xx}(x,y) &= (0,0,0), \quad \tilde{f}_{yy}(x,y) = (-x\sin y, -x\cos y, 0), \\ \tilde{f}_{xy}(x,y) &= \tilde{f}_{yx}(x,y) = (\cos y, -\sin y, 0), \\ \tilde{\nu} &= \frac{\tilde{f}_x \times \tilde{f}_y}{|\tilde{f}_x \times \tilde{f}_y|} = \frac{1}{\sqrt{1+x^2}}(\cos y, -\sin y, -x), \end{aligned}$$

and

$$(\tilde{g}_{ij}) = \left(\langle \tilde{f}_i, \tilde{f}_j \rangle\right) = \begin{pmatrix} 1 & 0 \\ 0 & 1+x^2 \end{pmatrix},$$

$$\left(\tilde{h}_{ij}\right) = \left(\langle \tilde{f}_{ij}, \tilde{\nu} \rangle\right) = \frac{1}{\sqrt{1+x^2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

from which

$$\tilde{K}(x,y) = \frac{\det(\tilde{h}_{ij})}{\det(\tilde{g}_{ij})} = \frac{-\frac{1}{1+x^2}}{1+x^2} = -\frac{1}{(1+x^2)^2},$$

so $K(x,y) = \tilde{K}(x,y)$.

b) We claim that f and $\tilde{f}$ are not locally isometric. Let $(x_0, y_0) \in \mathbb{R}_{\geq 0} \times \mathbb{R}$. Suppose there exist a neighborhood $U \subset \mathbb{R}_{\geq 0} \times \mathbb{R}$ of (x_0, y_0) and an isometry $\varphi\colon (U, g) \to (\varphi(U), \tilde{g})$. We write $\varphi(x,y) = (\tilde{x}(x,y), \tilde{y}(x,y))$. As the Gauss curvature is intrinsic we have $K(x,y) = \tilde{K}(\varphi(x,y)) = \tilde{K}(\tilde{x}, \tilde{y})$, that is

$$-\frac{1}{(1+x^2)^2} = -\frac{1}{(1+\tilde{x}^2)^2},$$

and hence $\tilde{x}(x,y) = x$, which implies that $\varphi(x,y) = (x, \tilde{y}(x,y))$.

Notice that the Gauss curvature is constant on curves with constant x-coordinate. Now consider a curve $\gamma(t) = (u(t), y_0)$ with $u(0) = x_0$, parametrized by arc length (with respect to g). The curve γ runs perpendicularly to curves with constant Gauss curvature (with respect to g). Its image $\tilde{\gamma} := \varphi \circ \gamma$ must also be parametrized by arc length and run perpendicularly to curves with constant Gauss curvature (with respect to $\tilde{g}$). Hence $\tilde{\gamma}(t) = (u(t), \tilde{y_0})$, where $\tilde{y_0} := \tilde{y}(x_0, y_0)$, and $|\dot{\tilde{\gamma}}(t)|_{\tilde{g}} = |\dot{u}(t)| = 1$, so $u(t) = x_0 \pm t$. Therefore we obtain

$$|\dot{\gamma}(t)|_g = \sqrt{1 + \frac{1}{u^2(t)}} \neq 1,$$

a contradiction to the fact that γ is parametrized by arc length with respect to g.

Alternative solution:

Now notice that since φ is an isometry it holds that the matrix of the first fundamental form of $\tilde{f}$ in $\varphi(x,y) = (x,\tilde{y})$ with respect to the basis $(\tilde{e}_1,\tilde{e}_2) := (d\varphi_{(x,y)}(e_1), d\varphi_{(x,y)}(e_2))$ of $T\tilde{U}_{x,\tilde{y}}$ ($\tilde{U} := \varphi(U)$) coincides with $(g_{ij}(x,y))$ and is therefore given by

$$\Big(\tilde{g}_{ij}(x,\tilde{y})\Big)_{(\tilde{e}_1,\tilde{e}_2)} = \begin{pmatrix} 1+\frac{1}{x^2} & 0 \\ 0 & x^2 \end{pmatrix}.$$

On the other hand the matrix of the first fundamental form of f with respect to the standard basis is given by $(\tilde{g}_{ij}(x,\tilde{y}))_{(e_1,e_2)} = (\tilde{g}_{ij}(x,\tilde{y}))$ and was computed in a).

The matrix of change of basis from (e_1,e_2) to $(\tilde{e}_1,\tilde{e}_2)$ is given exactly by

$$M := d\varphi_{(x,y)} = \begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix}$$

for some a,b. Thus it must hold that

$$\Big(\tilde{g}_{ij}(x,\tilde{y})\Big)_{(\tilde{e}_1,\tilde{e}_2)} = M^{\mathrm{T}} \cdot \Big(\tilde{g}_{ij}(x,\tilde{y})\Big)_{(e_1,e_2)} \cdot M,$$

which is equivalent to

$$\begin{cases} 1 + a^2(1+x^2) = 1 + \frac{1}{x^2}, \\ ab(1+x^2) = 0, \\ b^2(1+x^2) = x^2. \end{cases}$$

But the above system has no solutions for $x \in \mathbb{R}_{\geq 0}$. Contradiction arises.