Solution 6

1. Parallel Surfaces

Given an immersion $f\colon U \to \mathbb{R}^3$, $U \subset \mathbb{R}^2$, with Gauss map $\nu\colon U \to S^2 \subset \mathbb{R}^3$ and $\varepsilon > 0$ we define $f^\varepsilon\colon U \to \mathbb{R}^3$ as

$$f^\varepsilon(x_1, x_2) := f(x_1, x_2) + \varepsilon \cdot \nu(x_1, x_2).$$

Assuming that f has constant mean curvature $H \neq 0$ and non-vanishing Gauss curvature $K \neq 0$. Show that when $\varepsilon = \frac{1}{2H}$, f^ε is an immersion and the Gauss curvature of f^ε is constant.

Solution:

Fix $x \in U$ and choose an orthonormal basis (v_1, v_2) of (TU_x, g_x) consisting of eigenvectors of L_x. Since the Gauss curvature is invariant of reparametrization, we can assume without loss of generality that $(v_1, v_2) = (e_1, e_2)$ at x. Then $g_{ij} = \delta_{ij}$, $h_{ij} = \kappa_i \delta_{ij}$, $H = \frac{1}{2}(\kappa_1 + \kappa_2)$ and $K = \kappa_1\kappa_2$, where all these relations hold at x. Moreover $\nu_i = -\sum_{k=1}^{2} {h^k}_i \cdot f_k = -\kappa_i \cdot f_i$ and ν is a Gauss map for f^ε too.

It follows that, when evaluated in $x \in U$:

$$\begin{aligned}
f^\varepsilon_i &= f_i + \varepsilon \cdot \nu_i = (1 - \varepsilon\kappa_i) \cdot f_i,\\
f^\varepsilon_{ij} &= (1 - \varepsilon\kappa_i) \cdot f_{ij} - \varepsilon\kappa_{i,j} \cdot f_i,\\
g^\varepsilon_{ij} &= \langle f^\varepsilon_i, f^\varepsilon_j \rangle = (1 - \varepsilon\kappa_i)(1 - \varepsilon\kappa_j)\delta_{ij},\\
h^\varepsilon_{ij} &= \langle f^\varepsilon_{ij}, \nu \rangle = (1 - \varepsilon\kappa_i)\langle f_{ij}, \nu\rangle - \varepsilon\kappa_{i,j}\langle f_i, \nu\rangle = (1 - \varepsilon\kappa_i)\kappa_i\delta_{ij}.
\end{aligned}$$

When $\varepsilon = \frac{1}{2H}$, suppose f^ε is not regular at x. Then $0 = (1 - \varepsilon\kappa_1)(1 - \varepsilon\kappa_2) = 1 - 2H\varepsilon + \varepsilon^2 K(x) = \varepsilon^2 K(x)$, contradiction arises since $K(x) \neq 0$. Finally we obtain

$$K^\varepsilon(x) = \frac{\det(h^\varepsilon_{ij}(x))}{\det(g^\varepsilon_{ij}(x))} = \frac{(1 - \varepsilon\kappa_1)(1 - \varepsilon\kappa_2)\kappa_1\kappa_2}{(1 - \varepsilon\kappa_1)^2(1 - \varepsilon\kappa_2)^2} = \frac{K(x)}{1 - \varepsilon 2H + \varepsilon^2 K(x)},$$

and for $\varepsilon = \frac{1}{2H}$ is $K^\varepsilon = 4H^2$ constant.

2. Asymptotic Curves

Let $M \subset \mathbb{R}^3$ be a surface with $K < 0$. A curve $c\colon I \to M$ is called an *asymptotic curve* of M if $h_{c(t)}(\dot{c}(t), \dot{c}(t)) = 0$ for all $t \in I$. Prove that:

a) One can find a local parametrization of M whose parameter lines are asymptotic curves ("parametrization by asymptotic curves").

b) M is a minimal surface if and only if the asymptotic curves of a) are orthogonal to each other in every point.

Solution:

a) Let $p \in M$, $f\colon U \to M$ be a local parametrization with $f(0) = p$, $g_{ij}(0) = \delta_{ij}$ and $h_{ij}(0) = \kappa_i(0)\delta_{ij}$ as in Q1. We want to find vector fields $X_i\colon U' \subset U \to \mathbb{R}^2$ around 0 with $h_x(X_i(x), X_i(x)) = 0$ and

$X_1(x), X_2(x)$ linearly independent for $x \in U'$. For $X_i := (u_i, 1)$ it follows

$$h(X_i, X_i) = h_{11}u_i^2 + 2h_{12}u_i + h_{22},$$

so

$$h(X_i, X_i) = 0 \quad \Leftrightarrow \quad u_i = \frac{\pm\sqrt{h_{12}^2 - h_{11}h_{22}} - h_{12}}{h_{11}}.$$

Since $K(0) = h_{11}(0)h_{22}(0) < 0$, it holds that $h_{11} \neq 0$ and $h_{12}^2 - h_{11}h_{22} > 0$ in a neighborhood U' of 0 and the vector fields $X_1 = (u_1, 1), X_2 = (u_2, 1)$ with

$$u_1 = \frac{\sqrt{h_{12}^2 - h_{11}h_{22}} - h_{12}}{h_{11}} \quad \text{and} \quad u_2 = \frac{-\sqrt{h_{12}^2 - h_{11}h_{22}} - h_{12}}{h_{11}}$$

are well defined. Moreover $u_1 \neq u_2$ and therefore X_1, X_2 are linearly independent on U'. By Lemma A.5 there exists a diffeomorphism $\varphi\colon \tilde{U} \to \varphi(\tilde{U}) \subset U'$ with

$$\frac{\partial \varphi}{\partial x^i} = \lambda_i \cdot (X_i \circ \varphi)$$

in $\tilde{U}$ and $0 \in \varphi(\tilde{U})$, where $\lambda_i : \tilde{U} \to \mathbb{R}$. The map $\tilde{f}\colon \tilde{U} \to \mathbb{R}^3$, $\tilde{f} := f \circ \varphi$, is a parametrization with the desired properties.

Indeed, let $u_0 \in \mathbb{R}$ and consider the parameter curve $\gamma\colon I \to M$, with

$$\gamma(v) \coloneqq \tilde{f}(u_0, v) = \tilde{f} \circ \alpha(v),$$

provided that $\{u_0\} \times I \subset \tilde{U}$, where $\alpha(v) \coloneqq (u_0, v)$. Then

$$\begin{aligned}\dot{\gamma}(v) &= (f \circ \varphi \circ \alpha)'(v) = df_{\varphi(\alpha(v))}(\varphi \circ \alpha)'(v) \\ &= df_{\varphi(u_0,v)} \frac{\partial \varphi}{\partial v}(u_0, v) = \lambda_2(u_0, v) \cdot df_{\varphi(u_0,v)}\big(X_2(\varphi(u_0, v))\big),\end{aligned}$$

hence

$$\begin{aligned}&h_{\gamma(v)}\big(\dot{\gamma}(v), \dot{\gamma}(v)\big) \\ &\quad = \lambda_2(u_0, v)^2 \cdot h_{f(\varphi(u_0,v))}\Big(df_{\varphi(u_0,v)}(X_2(\varphi(u_0, v))), df_{\varphi(u_0,v)}(X_2(\varphi(u_0, v)))\Big) \\ &\quad = \lambda_2(u_0, v)^2 \cdot h_{\varphi(u_0,v)}\Big(X_2(\varphi(u_0, v)), X_2(\varphi(u_0, v))\Big) \\ &\quad = 0.\end{aligned}$$

Similarly, we have

$$h_{\tilde{\gamma}(u)}(\dot{\tilde{\gamma}}(u), \dot{\tilde{\gamma}}(u)) = 0$$

if we consider $\tilde{\gamma}(u) = \tilde{f}(u, v_0)$ for $(u, v_0) \in \tilde{U}$.

b) We choose a parametrization by asymptotic curves $f\colon U \to \mathbb{R}^3$, which exists by a). Then $h_{11} = h_{22} = 0$ and $h_{12} \neq 0$ (because $K \neq 0$).

For the mean curvature it holds that

$$H = \tfrac{1}{2}\,\mathrm{trace}(h^i{}_j) = \tfrac{1}{2} \cdot \sum_{i,j=1}^{2} g^{ij}h_{ji} = \tfrac{1}{2}\,\left(g^{12}h_{21} + g^{21}h_{12}\right) = g^{12}h_{12}$$

and hence

$$H = 0 \quad \Leftrightarrow \quad g^{12} = 0 \quad \Leftrightarrow \quad g_{12} = 0 \quad \Leftrightarrow \quad \langle f_1, f_2 \rangle = 0,$$

which implies that M is a minimal surface if and only if the parameter lines are orthogonal.

3. Conjugate Minimal Surfaces

Let $U \subset \mathbb{R}^2$ be an open set. Two isothermally parametrized minimal surfaces $f, \tilde{f}\colon U \to \mathbb{R}^3$ are called *conjugate* if $f_1 = \tilde{f}_2$ and $f_2 = -\tilde{f}_1$.

a) Find isothermal parametrizations of the helicoid and the catenoid and show that they are conjugate.

b) Show that if f and $\tilde{f}$ are conjugate then $\{f^t\colon U \to \mathbb{R}^3\}_{t\in\mathbb{R}}$ with

$$f^t(x) \coloneqq \cos t \cdot f(x) + \sin t \cdot \tilde{f}(x)$$

is a family of isothermally parametrized minimal surfaces.

c) Show that the surfaces f^t are locally isometric to each other and find a Gauss map for f^t.

Solution:

a) The catenoid is given by

$$\hat{f}(x, y) = (\cosh y \, \cos x, \, \cosh y \, \sin x, \, y).$$

We substitute $x \mapsto x + \frac{\pi}{2}$ and we obtain

$$f(x, y) = (\cosh y \, \sin x, -\cosh y \, \cos x, y).$$

Then

$$\begin{aligned} f_1(x, y) &= (\cosh y \, \cos x, \, \cosh y \, \sin x, \, 0), \\ f_2(x, y) &= (\sinh y \, \sin x, \, -\sinh y \, \cos x, \, 1). \end{aligned}$$

For the helicoid

$$\tilde{f}(x, y) = (\sinh y \, \cos x, \, \sinh y \, \sin x, \, x)$$

we obtain

$$\begin{aligned} \tilde{f}_1(x, y) &= (-\sinh y \, \sin x, \, \sinh y \, \cos x, \, 1), \\ \tilde{f}_2(x, y) &= (\cosh y \, \cos x, \, \cosh y \, \sin x, \, 0). \end{aligned}$$

Therefore

$$\begin{aligned} (g_{ij}) &= \Big(\langle f_i, f_j \rangle\Big) = \begin{pmatrix} \cosh^2 y & 0 \\ 0 & \sinh^2 y + 1 \end{pmatrix} = \cosh^2 y \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\ (\tilde{g}_{ij}) &= \Big(\langle \tilde{f}_i, \tilde{f}_j \rangle\Big) = \begin{pmatrix} \sinh^2 y + 1 & 0 \\ 0 & \cosh^2 y \end{pmatrix} = \cosh^2 y \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Define $\lambda(x,y) := \cosh y$, then it holds $g_{ij} = \tilde{g}_{ij} = \lambda^2\,\delta_{ij}$, $f_1 = \tilde{f}_2$ and $f_2 = -\tilde{f}_1$, from which we conclude that f and $\tilde{f}$ are isothermal and conjugate.

Notice that by Proposition 5.7 two conjugate isothermally parametrized surfaces are minimal. Indeed $f_{11} + f_{22} = \tilde{f}_{21} - \tilde{f}_{12} = 0$ and similarly $\tilde{f}_{11} + \tilde{f}_{22} = 0$.

b) We first show that f^t is an isothermal parametrization. Note that $\langle f_1, f_1\rangle = \langle \tilde{f}_2, \tilde{f}_2\rangle$, so we have $\langle f_i, f_j\rangle = \langle \tilde{f}_i, \tilde{f}_j\rangle = \lambda^2\delta_{ij}$ for some $\lambda : U \to \mathbb{R}$. Notice that

$$\begin{aligned}
\langle f_1, \tilde{f}_1\rangle + \langle f_1, \tilde{f}_1\rangle &= -2\cdot\langle f_1, f_2\rangle = 0,\\
\langle f_2, \tilde{f}_2\rangle + \langle f_2, \tilde{f}_2\rangle &= 2\cdot\langle f_1, f_2\rangle = 0,\\
\langle f_1, \tilde{f}_2\rangle + \langle f_2, \tilde{f}_1\rangle &= \langle f_1, f_1\rangle - \langle f_2, f_2\rangle = \lambda - \lambda = 0,
\end{aligned}$$

hence $\langle f_i, \tilde{f}_j\rangle + \langle f_j, \tilde{f}_i\rangle = 0$. It follows that

$$\begin{aligned}
g^t_{ij} = \langle f^t_i, f^t_j\rangle &= \cos^2 t\cdot\langle f_i, f_j\rangle + \cos t\,\sin t\cdot\Big(\langle f_i, \tilde{f}_j\rangle + \langle f_j, \tilde{f}_i\rangle\Big) + \sin^2 t\cdot\langle \tilde{f}_i, \tilde{f}_j\rangle\\
&= \lambda^2(\cos^2 t + \sin^2 t)\,\delta_{ij} = \lambda^2\,\delta_{ij},
\end{aligned}$$

so we conclude f^t is isothermal. Moreover

$$\begin{aligned}
\Delta f^t &= \cos t\,(f_{11} + f_{22}) + \sin t\,(\tilde{f}_{11} + \tilde{f}_{22})\\
&= \cos t\,(\tilde{f}_{21} - \tilde{f}_{12}) + \sin t\,(-f_{21} + f_{12}) = 0
\end{aligned}$$

and hence f^t is minimal by Proposition 5.7.

c) As computed in b), we have for all $t \in \mathbb{R}$

$$g^t_{ij} = \lambda^2\,\delta_{ij},$$

which is invariant of t, therefore the identity $\mathrm{id}\colon U \to U$ is an isometry between f^t for distinct t.

In order to find a Gauss map for f^t we compute

$$\begin{aligned}
f^t_1 \times f^t_2 &= \cos^2 t\,(f_1 \times f_2) + \cos t\,\sin t\,(f_1 \times \tilde{f}_2 + \tilde{f}_1 \times f_2) + \sin^2 t\,(\tilde{f}_1 \times \tilde{f}_2)\\
&= \cos^2 t\,(f_1 \times f_2) + \cos t\,\sin t\,(f_1 \times f_1 - f_2 \times f_2) + \sin^2 t\,(-f_2 \times f_1)\\
&= f_1 \times f_2 \quad (= \tilde{f}_1 \times \tilde{f}_2),
\end{aligned}$$

and therefore it holds that

$$\nu^t = \nu = \tilde{\nu} = \frac{f_1 \times f_2}{|f_1 \times f_2|}$$

for all $t \in \mathbb{R}$.