Solution 7

1. Characterization of the Sphere

Prove the following lemma due to H. Hopf:

Lemma. Let $m \geq 1$, M be a compact, connected, m-dimensional submanifold of $\mathbb{R}^{m+1}$. Suppose that for each vector $v \in S^m$ there exists $\lambda = \lambda(v) \in \mathbb{R}$ such that M is symmetric with respect to reflections on the hyperplane $E_{v,\lambda} := \{x \in \mathbb{R}^{m+1} : \langle x, v\rangle = \lambda\}$, then M is a sphere.
Hint: Show first that upon translation one can arrange that M is symmetric with respect to all coordinate hyperplanes and, hence, centrally symmetric with respect to the origin.

Solution:

First of all, notice that for $v \in S^m$ and $x \in \mathbb{R}^{m+1}$, $\langle x, v\rangle = \lambda(v)$ if and only if $\langle x - \lambda(v)v, v\rangle = 0$, hence $E_{v,\lambda(v)} = T_{\lambda(v)v}(E_{v,0})$ and in fact $E_{v,\lambda} = T_{\lambda v}(E_{v,0})$ for all $\lambda \in \mathbb{R}$, where $T_{\lambda v} \colon x \mapsto x + \lambda v$. Moreover the reflection on the hyperplane $E_{v,0}$ is given by $R_{v,0} \colon z \mapsto z - 2\langle z, v\rangle v$. Hence the reflection on the hyperplane $E_{v,\lambda(v)}$ is given by

$$R_{v,\lambda(v)} = T_{\lambda(v)v} \circ R_{v,0} \circ T_{-\lambda(v)v}, \quad z \mapsto z - 2\langle v, z - \lambda(v)v\rangle v.$$

Up to translating M, we might assume that M is symmetric with respect to the reflections on the hyperplanes $E_i := E_{e_i,0}$ (which is given by changing the sign of the i-th coordinate)[1]. By applying successively the reflections on the hyperplanes $E_1, \ldots, E_{m+1}$ we obtain that M is preserved by the map $x \mapsto -x$.

This implies that $\lambda(v) = 0$ for all $v \in S^m$. Indeed, first notice that if M is symmetric with respect to $E_{v,\lambda}$, then M is symmetric with respect to $E_{-v,\lambda}$. This follows because M is preserved by the map $x \mapsto -x$ and also $R_{-v,\lambda}(z) = -R_{v,\lambda}(-z)$. If $\lambda(v) \neq 0$ we can use subsequent reflections on the parallel hyperplanes $E_{v,\lambda(v)} \neq E_{-v,\lambda(v)}$ to produce an unbounded sequence of points in M. This is not possible by compactness, hence $\lambda(v) = 0$.

Now let $p \in M \setminus \{0\}$. For every point $q \in S_{|p|}(0)$, the sphere with radius $|p|$ around 0, there exists a reflection $R_{v,0}$ on the hyperplane $E_{v,0}$ (explicitly $v := \frac{q-p}{|q-p|}$) sending p to q, so $q \in M$. Hence $S_{|p|}(0)$ is contained in M. If M contains any other point p' with $|p'| \neq |p|$, then the same argument shows that M contains also the sphere of radius $|p'|$ and by connectedness also the region between the two spheres, contradicting the m-dimensionality of M.

Alternative Solution. By the separation theorem, we denote by A the bounded connected component in $\mathbb{R}^{m+1} \setminus M$ with $\partial A = M$. Let

$$p_0 := \frac{\int_A x \, d\mathcal{L}^{m+1}(x)}{\mathrm{Vol}(A)}$$

be the center of mass of A, where $\mathcal{L}^{m+1}$ is the Lebesgue measure on $\mathbb{R}^{m+1}$. By translation we can assume without loss of generality that $p_0 = 0$. Let $v \in S^m$, $\lambda \in \mathbb{R}$, and let $R_{v,\lambda}$ denote the reflection with respect to $E_{v,\lambda}$. Assume M is symmetric with respect to $R_{v,\lambda}$, then $R_{v,\lambda}(A)$ is the bounded connected component of $\mathbb{R}^{m+1} \setminus R_{v,\lambda}(M) = \mathbb{R}^{m+1} \setminus M$ since $R_{v,\lambda} : \mathbb{R}^{m+1} \to \mathbb{R}^{m+1}$ is a diffeomorphism, hence $R_{v,\lambda}(A) = A$. Moreover, since $R_{v,\lambda}$ is a bijective affine map, $R_{v,\lambda}(p_0)$ equals to the center of mass of $R_{v,\lambda}(A) = A$, hence $p_0 = R_{v,\lambda}(p_0)$. It follows that $0 = p_0 \in E_{v,\lambda}$ and hence $\lambda = 0$. By the assumptions in the problem, then we have M is symmetric with respect to $E_{v,0}$ for any $v \in S^m$. The remaining argument is the same as above.

[1] Denote by $T_1 : \mathbb{R}^{m+1} \to \mathbb{R}^{m+1}$ the translation by $-\lambda(e_1)\cdot e_1$ and show that $T_1(M)$ is symmetric with respect to reflections on the hyperplane $E_{e_1,0}$. Notice that $T_1(M)$ is still symmetric with respect to the hyperplanes $E_{e_i,\lambda(e_i)}$ for $i = 2, \dots, m+1$. Repeat for $T_2, \dots, T_{m+1}$.

2. Non-positively Curved Surfaces

Let $M \subset \mathbb{R}^3$ be a surface with Gauss curvature $K \leq 0$. Prove the following assertions (we assume $a < b$).

a) There is no simple geodesic loop (in particular no simple C^∞-closed geodesic) $c\colon [a,b] \to M$ whose trace bounds a topological disk in M.

b) There is no pair of injective geodesics $c_1, c_2\colon [a,b] \to M$ such that $c_1(a) = c_2(a)$ and $c_1(b) = c_2(b)$ are the only common points and the union of the traces bounds a topological disk.

c) If M is homeomorphic to a cylinder and $K < 0$, then there is no pair of simple C^∞-closed geodesics $c_1, c_2\colon [a,b] \to M$ with different traces.

Solution:

a) Suppose that there is a simple geodesic loop c bounding a simply connected region D homeomorphic to a disc. Denote by α the external angle in $c(a) = c(b)$. By the Gauss-Bonnet theorem (Theorem 6.3) we have

$$\int_D K\, dA + \alpha = 2\pi,$$

which is not possible as $K \leq 0$ and $\alpha \in (-\pi, \pi)$.

b) Suppose that the geodesics c_1 and c_2 enclose a compact simply connected region D homeomorphic to a disc. Denote by α_1, α_2 the external angles between them at $c_1(a)$ and $c_1(b)$ respectively. Since c_1 and c_2 are geodesics, it holds $\kappa_g = 0$. Hence by the Gauss-Bonnet theorem (Theorem 6.3) we have

$$\int_D K\, dA + \alpha_1 + \alpha_2 = 2\pi.$$

Since $K \leq 0$ it follows that $\alpha_1 + \alpha_2 \geq 2\pi$. Moreover, since $\alpha_1, \alpha_2 \in [-\pi, \pi]$ it must hold $\alpha_1 = \alpha_2 = \pi$. Therefore $\dot{c}_1(a) = \dot{c}_2(a)$ and thus by uniqueness of geodesics $c_1 = c_2$, a contradiction.

c) Suppose that there are two simple C^∞-closed geodesics c_1 and c_2 with different traces. From a) it follows that neither c_1 nor c_2 can enclose a disc, hence $M \setminus c_1([a,b])$ has two connected components homeomorphic to cylinders[2]. Also, c_1 and c_2 cannot intersect. Otherwise assume they intersect at p, then by uniqueness of geodesics c_2 cannot be tangent to c_1 at p, hence c_2 intersects with both connected components of $M \setminus c_1([a,b])$, and must have another intersection point $q \neq p$ with c_1. Then (by Schoenflies theorem) c_1 and c_2 enclose a topological disk, contradiction arises. Therefore they must be disjoint and enclose an annulus.

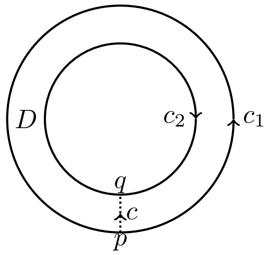
We parametrize c_1, c_2 such that their orientation coincides with the one of the domain that they enclose. Choose two points p on c_1 and q on c_2 and a simple regular smooth curve c connecting p to q, which doesn't intersect c_1 and c_2 at any other point. Then the concatenation $c_1 \cup c \cup c_2 \cup -c$ encloses a compact simply connected region D homeomorphic to a disc, where $-c$ is along the reverse direction of c[3]. It holds

$$\int_{c_1 \cup c \cup c_2 \cup -c} \kappa_g = \int_{c_1} \kappa_g + \int_c \kappa_g + \int_{c_2} \kappa_g + \int_{-c} \kappa_g = \int_c \kappa_g - \int_c \kappa_g = 0.$$

Since the external angles sum to 2π, the Gauss-Bonnet theorem gives

$$0 > \int_D K\,dA = 2\pi - \sum_{i=1}^{4} \alpha_i = 0,$$

which gives the desired contradiction.



Alternative Solution. Suppose the simple C^∞-closed geodesics c_1 and c_2 don't intersect, then they enclose a compact annulus R as pointed above with $\chi(R) = 0$. By the generalized Gauss-Bonnet theorem as mentioned in the lecture, we have

$$0 = 2\pi\chi(R) = \int_R K\,dA + \int_{\partial R} \kappa_g\,ds = \int_R K\,dA < 0,$$

contradiction arises.

[2]Rigorously, c_1 can be identified as a simple closed continuous curve in $\mathbb{R}^2$, where $0 \in \mathbb{R}^2$ corresponds to the bottom and ∞ in $\mathbb{R}^2$ corresponds to the top of the cylinder. Then we can apply Schoenflies theorem.

[3]In rigorous argument, it's replaced by a simple regular smooth curve γ from $q' \in c_2([a,b])$ to $p' \in c_1([a,b])$, which doesn't intersect c_1 and c_2 at any other point and is disjoint from c. We can choose γ to be as close to $-c$ as possible.

3. Gauss Map of the Torus

a) Let $f\colon U \to \mathbb{R}^{m+1}$, $U \subset \mathbb{R}^m$ open, be an immersion with Gauss map $\nu\colon U \to S^m \subset \mathbb{R}^{m+1}$. Assuming that ν is an immersion, prove that

$$A(\nu) = \int_U |K| \sqrt{\det(g_{ij})}\,dx.$$

b) Let $T \subset \mathbb{R}^3$ be a torus. Describe the image of the Gauss map and prove that

$$\int_T K\,dA = 0,$$

without using the theorem of Gauss-Bonnet.

Solution:

a) From Weingarten's equation (Lemma 4.8 (2)) it follows that

$$\nu_i = -\sum_{k=1}^{m} h^k{}_i f_k,$$

and hence

$$\begin{aligned}\langle \nu_i, \nu_j \rangle &= \left\langle -\sum_{k=1}^{m} h^k{}_i f_k, -\sum_{l=1}^{m} h^l{}_j f_l \right\rangle \\ &= \sum_{k=1}^{m}\sum_{l=1}^{m} h^k{}_i h^l{}_j \langle f_k, f_l \rangle \\ &= \sum_{k=1}^{m}\sum_{l=1}^{m} h^k{}_i h^l{}_j g_{kl} \\ &= \sum_{k=1}^{m}\sum_{l=1}^{m} h^k{}_i g_{kl} h^l{}_j .\end{aligned}$$

So we get

$$\begin{aligned}\det\left(\langle \nu_i, \nu_j \rangle\right) &= \det\left(\left(h^k{}_i\right)_{ik} \circ \left(g_{kl}\right)_{kl} \circ \left(h^l{}_j\right)_{lj}\right) \\ &= \det(h^k{}_i) \cdot \det(h^l{}_j) \cdot \det(g_{kl}) \\ &= K^2 \cdot \det(g_{kl})\end{aligned}$$

and therefore

$$A(\nu) = \int_U \sqrt{\det\langle \nu_i, \nu_j \rangle}\, dx = \int_U |K| \sqrt{\det(g_{ij})}\, dx.$$

b) The image of the Gauss map covers the whole sphere S^2. The circles $(r\cos y, r\sin y, \pm a)$ are mapped by the Gauss map ν to the South and North poles, respectively.

For $q \in S^2$ distinct from the South and North poles, there are exactly two points p_+ and p_- in T such that $\nu(p_+) = \nu(p_-) = q$, one lying in the outer region T_+ with $K > 0$ and one lying in the inner region T_- with $K < 0$ (see also later). Therefore

$$\int_T K\, dA = \int_{T_+} K\, dA + \int_{T_-} K\, dA = A(\nu_+) - A(\nu_-) = A(S^2) - A(S^2) = 0.$$

Alternative Solution. The parametrization of the torus is given by $f\colon [0, 2\pi]^2 \to \mathbb{R}^3$ with

$$f(x, y) = \left((r + a\cos x)\cos y, (r + a\cos x)\sin y, a\sin x\right),$$

where $r > a > 0$. It holds

$$\begin{aligned}f_1(x, y) &= \left(-a\sin x\cos y, -a\sin x\sin y, a\cos x\right), \\ f_2(x, y) &= \left(-(r + a\cos x)\sin y, (r + a\cos x)\cos y, 0\right), \\ f_{11}(x, y) &= \left(-a\cos x\cos y, -a\cos x\sin y, -a\sin x\right), \\ f_{12}(x, y) &= \left(a\sin x\sin y, -a\sin x\cos y, 0\right), \\ f_{22}(x, y) &= \left(-(r + a\cos x)\cos y, -(r + a\cos x)\sin y, 0\right)\end{aligned}$$

and

$$\nu(x, y) = \frac{f_1 \times f_2}{|f_1 \times f_2|} = (-\cos x\cos y, -\cos x\sin y, -\sin x).$$

From the above computations we obtain

$$(g_{ij}) = \begin{pmatrix} a^2 & 0 \\ 0 & (r + a\cos x)^2 \end{pmatrix},$$

$$(h_{ij}) = \begin{pmatrix} a & 0 \\ 0 & (r + a\cos x)\cos x \end{pmatrix}.$$

For the Gauss curvature it holds

$$K = \frac{\det(h_{ij})}{\det(g_{ij})} = \frac{a(r + a\cos x)\cos x}{a^2(r + a\cos x)^2} = \frac{\cos x}{a(r + a\cos x)},$$

and therefore

$$\int_T K\,dA = \int_0^{2\pi}\int_0^{2\pi} \frac{\cos x}{a(r + a\cos x)}\sqrt{\det(g_{ij})}\,dx\,dy = 2\pi\int_0^{2\pi}\cos x\,dx = 0.$$