Solution 8

1. The Brouwer Fixed Point Theorem

Brouwer's theorem states that every continuous self-map $f\colon D \to D$ of the unit ball $D := \{x \in \mathbb{R}^n : |x| \leq 1\}$ has a fixed point.

a) Let $M \subset \mathbb{R}^3$ be a surface and $\tilde{D} \subset M$ a region diffeomorphic to the disc $D := \{x \in \mathbb{R}^2 : |x| \leq 1\}$. Consider a continuous tangent vector field $X\colon \tilde{D} \to \mathbb{R}^3$ which on $\partial\tilde{D}$ is pointing outward. Show that X has zeros in the interior of $\tilde{D}$.

b) Prove the Brouwer fixed point theorem in two dimensions using part a).

Solution:

a) It suffices to prove the statement for $X\colon D \to \mathbb{R}^2$.

We want to use the Poincaré index theorem, but for that we must have a compact surface without boundary.

First we can modify X such that on ∂D it points radially towards the exterior[1]. Then we consider $Y\colon D \to \mathbb{R}^2$, $Y := -X$, which is a vector field on D pointing radially towards the interior at every point of ∂D.

Now identify D with a hemisphere of S^2, then we can glue two hemispheres together along their boundaries to obtain S^2. By considering the vector field X on one hemisphere and Y on the other we obtain a continuous vector field $Z\colon S^2 \to \mathbb{R}^3$, which is nowhere vanishing on the equator. As seen in class, the Poincaré index theorem implies that Z must have a zero, but since there are none on the equator we conclude that X or $-X$ (and hence X) must have at least one zero in the interior of D.

b) Let $f\colon D \to D$ be a continuous map. We define the vector field $X\colon D \to \mathbb{R}^2$ by $X(x) := x - f(x)$. For $x \in \partial D$ it holds

$$\langle X(x), x\rangle = \langle x, x\rangle - \langle f(x), x\rangle \geq 1 - |x|\,|f(x)| \geq 0,$$

where the equality holds if and only if $f(x) = x$. This shows that if X doesn't vanish on ∂D, then it points outward at every point of ∂D. In this case it follows from *a)* that it has a zero in the interior of D. In both cases there is $x_0 \in D$ with $X(x_0) = 0$, that is $f(x_0) = x_0$.

[1] To be specific, assume X points outward on ∂D and doesn't have zeros in D°, then by uniform continuity there exists $r \in (0,1)$ such that $\langle X(x), x\rangle > 0$ on $D \setminus D_r^\circ$, where D_r is the disk centered at 0 of radius r. We define $\tilde{X} = X$ on D_r and $\tilde{X}(x) := \frac{1-|x|}{1-r}X(x) + (|x| - r)x$ if $|x| \in [r, 1]$. Then $\tilde{X}$ points radially towards the exterior on ∂D and is nowhere vanishing on D (note $\langle \tilde{X}(x), x\rangle > 0$ if $r \leq |x| \leq 1$), and we can obtain a contradiction as in the solution.

2. Hyperbolic Trigonometry

Consider a geodesic triangle with angles α, β, γ at the vertices A, B, C and sides of lengths a, b, c opposite to A, B, C, respectively, in the hyperbolic plane $(H^2, g) \subset \mathbb{R}^{2,1}$. Prove the following trigonometric identities of hyperbolic geometry:

a) $\sinh c \, \sin\beta = \sinh b \, \sin\gamma$ (law of sines),

b) $\cosh c = \cosh a \, \cosh b - \sinh a \, \sinh b \, \cos\gamma$ (law of cosines),

c) $\cos\gamma = \sin\alpha \, \sin\beta \, \cosh c - \cos\alpha \, \cos\beta$ (law of cosines for angles).

Hint: Choose B in e_3 and C in the plane spanned by e_1 and e_3, then compute the coordinates of A in two different ways.

Solution:

We denote by σ_{PQ} the unit-speed geodesic from P to Q and by $v_{PQ} := \sigma'_{PQ}(0) \in TH^2_P$ the vector in direction of Q. We know that

$$\sigma_{PQ}(s) = \cosh(s)\, P + \sinh(s)\, v_{PQ}.$$

We assume that $B = e_3$, $v_{BC} = e_1$, and $g(v_{BA}, e_2) \geq 0$. Then

$$\begin{aligned} A &= \cosh c \cdot B + \sinh c \cdot v_{BA} \\ &= \cosh c \cdot e_3 + \sinh c \cdot (\cos\beta \cdot e_1 + \sin\beta \cdot e_2). \end{aligned}$$

On the other hand,

$$\begin{aligned} A &= \cosh b \cdot C + \sinh b \cdot v_{CA} \\ &= \cosh b \cdot (\cosh a \cdot e_3 + \sinh a \cdot e_1) + \sinh b \cdot v_{CA}. \end{aligned}$$

To determine v_{CA}, notice that $v_{CA} \in TH^2_C = \operatorname{span}\{v_{CB}, e_2\}$ and

$$v_{CB} = -\sigma'_{BC}(a) = -(\sinh a \cdot e_3 + \cosh a \cdot e_1).$$

Since (v_{CB}, e_2) is an orthonormal basis of TH^2_C, and since $g(v_{CB}, v_{CA}) = \cos\gamma$ and $g(e_2, v_{CA}) \geq 0$, it follows that

$$\begin{aligned} v_{CA} &= \cos\gamma \cdot v_{CB} + \sin\gamma \cdot e_2 \\ &= -\cos\gamma \cdot (\sinh a \cdot e_3 + \cosh a \cdot e_1) + \sin\gamma \cdot e_2. \end{aligned}$$

We conclude that

$$A = \begin{pmatrix} \sinh c \, \cos\beta \\ \sinh c \, \sin\beta \\ \cosh c \end{pmatrix} = \begin{pmatrix} \sinh a \, \cosh b - \cosh a \, \sinh b \, \cos\gamma \\ \sinh b \, \sin\gamma \\ \cosh a \, \cosh b - \sinh a \, \sinh b \, \cos\gamma \end{pmatrix}.$$

This yields a) and b) as well as a third identity. Using the latter and the law of sines, with permuted entries, we get

$$\begin{aligned} &\sinh a\,(\sin\alpha \, \sin\beta \, \cosh c - \cos\alpha \, \cos\beta) \\ &\quad = (\sinh a \, \sin\beta) \sin\alpha \, \cosh c - (\sinh a \, \cos\beta) \cos\alpha \\ &\quad = (\sinh b \, \sin\alpha) \sin\alpha \, \cosh c - (\sinh c \, \cosh b - \cosh c \, \sinh b \, \cos\alpha) \cos\alpha \\ &\quad = \sinh b \, \cosh c - \cosh b \, \sinh c \, \cos\alpha \\ &\quad = \sinh a \, \cos\gamma. \end{aligned}$$

This gives c). Note that this shows explicitly that in hyperbolic geometry (unlike in Euclidean geometry), the three angles of a geodesic triangle determine its side lengths.

3. Circles in H^2

Consider a circle $S \subset H^2$ of radius $r > 0$ in the hyperbolic plane. Determine the length L of S, the (constant) geodesic curvature κ of S (with respect to the inward normal), and the area A of the disk $D \subset H^2$ bounded by S. (Hint: save work by making use of a relation between these three quantities.) What is the behavior of $\kappa = \kappa(r)$ when $r \to \infty$?

Solution:

First, recall that by the homogeneity of the hyperbolic plane (Theorem 7.2), L, A, κ do not depend on the position of the center of the circle (needless to say that all quantities are measured with respect to the hyperbolic metric, independent of a specific model – otherwise the problem would not be well-posed). Suppose that $p = (0,0,1) \in H^2 \subset \mathbb{R}^{2,1}$ is the center of S. For any unit vector $v \in TH^2_p$, the geodesic c with $c(0) = p$ and $c'(0) = v$ is given by $c(s) = \cosh(s)p + \sinh(s)v$. Hence, S is the circle of Euclidean radius $\sinh(r)$ in the horizontal plane $P = \mathbb{R}^2 \times \{\cosh(r)\} \subset \mathbb{R}^{2,1}$. Since $\langle w, w\rangle_{\mathrm{L}} = |w|^2$ (Euclidean norm) for all vectors w tangent to P, it follows that $L = 2\pi \sinh(r)$. The area is

$$A = \int_0^r 2\pi \sinh(s)\, ds = 2\pi(\cosh(r) - 1).$$

By the Gauss–Bonnet theorem, $-A + \kappa L = 2\pi$, thus $\kappa = \kappa(r) = \coth(r)$, and $\kappa(r) \to 1$ for $r \to \infty$. [By contrast, in $\mathbb{R}^2$, $\kappa(r) = \frac{1}{r} \to 0$.]

Alternative 1. For S as above and $\rho := \sinh(r)$, a unit speed parametrization of S is given by

$$t \mapsto \sigma(t) := (\cos(\rho^{-1}t)\rho, \sin(\rho^{-1}t)\rho, \cosh(r)).$$

Then $\sigma''(0) = (-\rho^{-1}, 0, 0)$, and the inward unit normal of S at $\sigma(0)$ is the vector $-c'(r) = -(\cosh(r), 0, \rho)$, where $s \mapsto c(s) = (\sinh(s), 0, \cosh(s))$ is the radial geodesic through $\sigma(0)$. Hence,

$$\kappa = \langle \sigma''(0), -c'(r)\rangle_{\mathrm{L}} = \rho^{-1}\cosh(r) = \coth(r).$$

Alternative 2. Let S be the circle with center $p = 0$ and (hyperbolic) radius r in the Poincaré (disk) model (U, g) of H^2, where $g_x = 4(1 - |x|^2)^{-2}\langle \cdot\,, \cdot\rangle$. If $a \in (0,1)$ is the Euclidean radius of S, then

$$r = \int_0^a \frac{2}{1-t^2}\, dt = \int_0^a \frac{1}{1+t} + \frac{1}{1-t}\, dt = \log\left(\frac{1+a}{1-a}\right),$$

thus $a = (e^r - 1)/(e^r + 1)$ [$= \tanh(r/2)$] and

$$L = 2\pi a \cdot \frac{2}{1-a^2} = \ldots = 2\pi \sinh(r).$$

Warning. Here, the upper halfspace model (U^+, g^+) is a bad choice. This model is conformal, thus hyperbolic circles are still represented by Euclidean circles; however, the respective centers do not coincide. Recall that $s \mapsto (0, e^s)$ is a unit speed geodesic. Hence, the hyperbolic circle with center $p = (0,1)$ and radius $r > 0$ passes through the points $(0, e^{-r})$ and $(0, e^r)$, whose Euclidean midpoint is $(0, \cosh(r))$.