Solution 9

1. Isometries of the Hyperbolic Upper Half-Plane

Let $H^2 := \{x+iy \in \mathbb{C} : y > 0\}$ be the upper half-plane endowed with the hyperbolic metric $(g_{ij})(x,y) = \frac{1}{y^2}(\delta_{ij})$. The group $\mathrm{GL}(2,\mathbb{R})$ acts on H^2 in the following way.

Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2,\mathbb{R})$, then the group action is given by

$$z \longmapsto \frac{az+b}{cz+d}, \qquad \text{if } ad-bc > 0,$$
$$z \longmapsto \frac{a\bar{z}+b}{c\bar{z}+d}, \qquad \text{if } ad-bc < 0.$$

a) Prove that $\mathrm{GL}(2,\mathbb{R})$ acts on H^2 by isometries.

b) Prove that the isometry group of H^2 is isomorphic to $\mathrm{PGL}(2,\mathbb{R}) := \mathrm{GL}(2,\mathbb{R})/\{\lambda I : \lambda \neq 0\}$, where I is the identity in $\mathrm{GL}(2,\mathbb{R})$.

Solution:

a) We first show that $\mathrm{GL}(2,\mathbb{R})$ acts on H^2:
For $z = x+iy$ with $y > 0$ and $ad-bc > 0$, it follows that

$$\frac{az+b}{cz+d} = \frac{(az+b)(c\bar{z}+d)}{|cz+d|^2} = \frac{ac|z|^2 + bd + bc\bar{z} + adz}{|cz+d|^2},$$

so

$$\Im\left(\frac{az+b}{cz+d}\right) = \frac{(ad-bc)y}{|cz+d|^2} > 0,$$

and for $z = x+iy$ with $y > 0$ and $ad-bc < 0$ it follows that

$$\Im\left(\frac{a\bar{z}+b}{c\bar{z}+d}\right) = \frac{(ad-bc)(-y)}{|cz+d|^2} > 0.$$

As for compatibility, we compute

$$\frac{a_1\frac{a_2z+b_2}{c_2z+d_2}+b_1}{c_1\frac{a_2z+b_2}{c_2z+d_2}+d_1} = \frac{(a_1a_2+b_1c_2)z+(a_1b_2+b_1d_2)}{(c_1a_2+d_1c_2)z+(c_1b_2+d_2d_1)},$$

which is $\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$ acting at z, and the other cases are similar. We now show that it's an action by isometries. Notice that

$$\frac{az+b}{cz+d} = \frac{a}{c} - \frac{ad-bc}{c^2z+cd}$$

and therefore we can write every action as composition of maps of the form $f_1 : z \mapsto az$ ($a > 0$), $f_2 : z \mapsto z+b$ ($b \in \mathbb{R}$), $f_3 : z \mapsto -\bar{z}$ and $f_4 : z \mapsto -\frac{1}{z}$. These are all isometries. We show the computations for f_4:

Let $u := 1$, $v := i \in TM_z$, then $f_4(z) = -\frac{\overline{z}}{|z|^2}$ and $f_4'(z) = \frac{1}{z^2} = \frac{\overline{z}^2}{|z|^4}$. Therefore

$$g_{f_4(z)}(df_4(u), df_4(u)) = \frac{|z|^4}{y^2}\left\langle \frac{\overline{z}^2}{|z|^4}, \frac{\overline{z}^2}{|z|^4}\right\rangle = \frac{|z|^4}{y^2}\frac{|\overline{z}^2|^2}{|z|^8} = \frac{1}{y^2} = g_z(u,u),$$
$$g_{f_4(z)}(df_4(v), df_4(v)) = \frac{|z|^4}{y^2}\left\langle \frac{\overline{z}^2 i}{|z|^4}, \frac{\overline{z}^2 i}{|z|^4}\right\rangle = \frac{|z|^4}{y^2}\frac{|\overline{z}^2 i|^2}{|z|^8} = \frac{1}{y^2} = g_z(v,v),$$
$$g_{f_4(z)}(df_4(u), df_4(v)) = \frac{|z|^4}{y^2}\left\langle \frac{\overline{z}^2}{|z|^4}, \frac{\overline{z}^2 i}{|z|^4}\right\rangle = 0.$$

b) The group action of $\mathrm{GL}(2,\mathbb{R})$ is transitive on points and on their tangent spaces. Indeed,

$$x + iy = \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \cdot i,$$

moreover for $v \in S^1 \subset TH^2_i = \mathbb{C}$ we can find a point $a + bi \in S^1$ with $v = (a+bi)^2$. We define $f(z) := \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \cdot z$. Then on one hand

$$f(i) = \frac{ai + b}{-bi + a} = i$$

while on the other hand

$$df_i(1) = \frac{a^2+b^2}{(bi-a)^2} = \frac{(a+bi)^2}{a^2+b^2} = v.$$

With $z \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \cdot z = -\overline{z}$ we obtain also all orientation-reversing isometries of H^2.

As an isometry of H^2 is completely determined by the image of one point and one orthonormal basis of the tangent space at that point (since isometries preserve geodesics), we have found a surjective group homomorphism $\Phi\colon \mathrm{GL}(2,\mathbb{R}) \to \mathrm{Isom}(H^2)$. We will now compute the kernel of Φ.

The matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is in the kernel of Φ if and only if, for all $z \in H^2$ it holds that

$$\frac{az+b}{cz+d} = z \quad\Leftrightarrow\quad cz^2 + (d-a)z - b = 0 \quad\Leftrightarrow\quad a = d \text{ and } b = c = 0.$$

We conclude that $\operatorname{Ker}\Phi = \{\lambda I : \lambda \neq 0\}$ and hence

$$\mathrm{Isom}(H^2) \cong \mathrm{GL}(2,\mathbb{R})/\{\lambda I : \lambda \neq 0\} = \mathrm{PGL}(2,\mathbb{R}).$$

2. Differentiable Structures on $\mathbb{R}$

a) Let $\varphi, \psi\colon \mathbb{R} \to \mathbb{R}$ be defined as

$$\varphi(x) := x^3 \quad \text{and} \quad \psi(x) := \begin{cases} x, & x < 0, \\ 2x, & x \geq 0. \end{cases}$$

Do the atlases $\mathcal{A}_1 := \{\mathrm{id}_\mathbb{R}\}$, $\mathcal{A}_2 := \{\varphi\}$ and $\mathcal{A}_3 := \{\psi\}$ induce different differentiable structures on $\mathbb{R}$?

b) Prove that if two C^∞-manifolds are homeomorphic to $\mathbb{R}$, then they are diffeomorphic to each other.

Hint: Prove that every differentiable structure on $\mathbb{R}$ is diffeomorphic to the one induced by $\mathcal{A}_1$.

Solution:

a) The three atlases induce different C^∞-structures on $\mathbb{R}$ as the identity map $\mathrm{i}_\mathbb{R}$ from $(\mathbb{R}, \mathcal{A}_i)$ to $(\mathbb{R}, \mathcal{A}_j)$, $i \neq j$, is not a diffeomorphism (i.e. the charts are not C^∞-compatible).

b) Let $(M, \mathcal{A})$ be a C^∞-manifold and $\phi\colon M \to \mathbb{R}$ a homeomorphism. Then

$$\mathcal{A}' := \{(\phi(U), \psi \circ \phi^{-1}) : (U, \psi) \in \mathcal{A}\}$$

is a C^∞-atlas on $\mathbb{R}$ and $\phi\colon (M, \mathcal{A}) \to (\mathbb{R}, \mathcal{A}')$ is a diffeomorphism. We will now show that $(\mathbb{R}, \mathcal{A}')$ and $(\mathbb{R}, \mathcal{A}_1)$ are diffeomorphic. By transitivity this will show that that any two smooth manifolds homeomorphic to $\mathbb{R}$ are diffeomorphic to each other.

We will first show that there exists a (global) chart $(U', \psi') \in \mathcal{A}'$ with $U' = \mathbb{R}$. Choose charts $\{(U_i', \psi_i')\}_{i \in \mathbb{Z}} \subset \mathcal{A}'$ with U_i' connected, such that $\{U_i'\}_{i \in \mathbb{Z}}$ is a covering of $\mathbb{R}$. We can assume that $U_i' = (a_i, b_i)$ and $a_i < b_{i-1} < a_{i+1} < b_i$. We can also assume[1] that the maps ψ_i''s are strictly monotonically increasing and $\psi_i'((a_i, b_i)) = (c_i, d_i)$ with $d_i < c_{i+1}$, $c_i \to \infty$ $(i \to \infty)$ and $c_i \to -\infty$ $(i \to -\infty)$. In particular the ψ_i''s have the same orientation.

Now, let $\{\lambda_i \colon \mathbb{R} \to [0,1]\}_{i \in \mathbb{Z}}$ be a subordinate C^∞-partition of unity. Notice that it can be chosen such than $\lambda_i|_{(a_i, b_{i-1})}$ is monotonically increasing and $\lambda_i|_{(a_{i+1}, b_i)}$ is monotonically decreasing.

We define the map $\psi' \colon \mathbb{R} \to \mathbb{R}$,

$$\psi'(t) := \sum_{i \in \mathbb{Z}} \lambda_i(t) \cdot \psi_i'(t).$$

This map is strictly monotone, since for $s < t \in U_i' \cap U_{i+1}'$ it holds that

$$\begin{aligned}
\psi'(s) &= \lambda_i(s)\psi_i'(s) + (1 - \lambda_i(s))\psi_{i+1}'(s) && < \lambda_i(s)\psi_i'(t) + (1 - \lambda_i(s))\psi_{i+1}'(t) \\
&= \psi_{i+1}'(t) + \underbrace{\lambda_i(s)}_{\geq \lambda_i(t)} \underbrace{(\psi_i'(t) - \psi_{i+1}'(t))}_{<0} && \leq \psi_{i+1}'(t) + \lambda_i(t)(\psi_i'(t) - \psi_{i+1}'(t)) \\
&= \lambda_i(t)\psi_i'(t) + (1 - \lambda_i(t))\psi_{i+1}'(t) && = \psi'(t).
\end{aligned}$$

Hence ψ' is a homeomorphism. Moreover for a chart $(U'', \psi'') \in \mathcal{A}'$ we have

$$\psi' \circ \psi''^{-1} = \sum_{i \in \mathbb{Z}} (\lambda_i \circ \psi''^{-1}) \cdot (\psi_i' \circ \psi''^{-1}) \in C^\infty(\mathbb{R}),$$

since the λ_i's are smooth and the $(\psi_i' \circ \psi''^{-1})$'s are change of coordinates in $\mathcal{A}'$. If now (U'', ψ'') is positively oriented[2], we also have

$$\tfrac{d}{dt}(\psi_i' \circ \psi''^{-1})(t) > 0.$$

So for any $t \in \psi''(U_i' \cap U_{i+1}' \cap U'')$, there exists $\varepsilon > 0$ (possibly depending on t) such that

$$\tfrac{d}{dt}(\psi_i' \circ \psi''^{-1})(t),\ \tfrac{d}{dt}(\psi_{i+1}' \circ \psi''^{-1})(t) \geq \varepsilon.$$

Then

$$\begin{aligned}
&\frac{d}{dt}\Big(\psi' \circ \psi''^{-1}\Big)(t) \\
&\quad= \frac{d}{dt}\Big((\lambda_i \circ \psi''^{-1})(t) \cdot (\psi'_i \circ \psi''^{-1})(t) + (1 - (\lambda_i \circ \psi''^{-1})(t)) \cdot (\psi'_{i+1} \circ \psi''^{-1})(t)\Big) \\
&\quad= \underbrace{\tfrac{d}{dt}(\lambda_i \circ \psi''^{-1})(t)}_{\leq 0} \cdot \underbrace{\big((\psi'_i \circ \psi''^{-1})(t) - (\psi'_{i+1} \circ \psi''^{-1})(t)\big)}_{\leq 0} \\
&\qquad + (\lambda_i \circ \psi''^{-1})(t) \cdot \underbrace{\tfrac{d}{dt}(\psi'_i \circ \psi''^{-1})(t)}_{\geq \varepsilon} \\
&\qquad + (1 - (\lambda_i \circ \psi''^{-1})(t)) \cdot \underbrace{\tfrac{d}{dt}(\psi'_{i+1} \circ \psi''^{-1})(t)}_{\geq \varepsilon} \\
&\quad\geq (\lambda_i \circ \psi''^{-1})(t) \cdot \varepsilon + (1 - (\lambda_i \circ \psi''^{-1})(t)) \cdot \varepsilon = \varepsilon > 0.
\end{aligned}$$

Therefore by the inverse function theorem $\psi'' \circ \psi'^{-1} = (\psi' \circ \psi''^{-1})^{-1}$ is smooth and so $(\mathbb{R}, \psi') \in \mathcal{A}'$.

Now set $\Psi\colon (\mathbb{R}, \mathcal{A}') \to (\mathbb{R}, \mathcal{A}_1)$, $\Psi(t) := \psi'(t)$. For a chart $(U'', \psi'') \in \mathcal{A}'$ it holds that

$$\mathrm{id}_{\mathbb{R}} \circ \Psi \circ \psi''^{-1} = \psi' \circ \psi''^{-1}$$

is a diffeomorphism, because it's a change of coordinates in $\mathcal{A}'$.

This shows that Ψ is a diffeomorphism between $(\mathbb{R}, \mathcal{A}')$ and $(\mathbb{R}, \mathcal{A}_1)$.

[1] Up to taking smaller intervals and composing with the multiplication by -1 or translations in the image.

[2] Otherwise make it positively oriented by composing it with the multiplication by -1 in the image. If we prove that the positively oriented one is a diffeomorphism then also the negatively oriented one will be.

3. Proper Functions

A function $f\colon M \to \mathbb{R}$ is called *proper* if $f^{-1}(K)$ is compact for every compact subset $K \subset \mathbb{R}$.

Let M be a C^∞-manifold. Prove that there exists a proper C^∞-function on M.

Hint: Use a partition of unity.

Solution:

Let $\{U_i\}_{i=1}^\infty$ be an open covering of M, such that $\overline{U_i}$ is compact. Let $\{\lambda_i\colon M \to [0,1]\}_{i=1}^\infty$ be a subordinate C^∞ partition of unity. The we define $f\colon M \to \mathbb{R}$ by

$$f(p) := \sum_{i=1}^{\infty} i\lambda_i(p).$$

As the sum is locally finite, it follows that $f \in C^\infty(M)$.

Now, let $K \subset [-N, N]$ be a compact set. We claim that $f^{-1}(K) \subset \bigcup_{i=1}^N \overline{U_i}$ Indeed, if $p \notin \bigcup_{i=1}^N \overline{U_i}$, then $\lambda_i(p) = 0$ for $i = 1, \dots, N$ and therefore

$$f(p) = \sum_{i=N+1}^{\infty} i\lambda_i(p) > \sum_{i=N+1}^{\infty} N\lambda_i(p) = N\sum_{i=1}^{\infty} \lambda_i(p) = N.$$

This proves the claim and since $f^{-1}(K)$ is a closed subset of the compact set $\bigcup_{i=1}^N \overline{U_i}$, is itself compact.