D–MATH

Exam Probability Theory

01-3601-00S

Last Name *First Name* *Legi-Nr.*

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Please take note of the information on the answer-booklet.

Question 1

[11 Points] Let $(X_i)_{i\geq 1}$ be a sequence of i.i.d. random variables following the uniform distribution on $[0, 1]$.

(1) **[3 Points]** State the Borel-Cantelli lemmas.

(2) **[3 Points]** Fix $\varepsilon \in (0, 1)$. Show that almost surely there exists an integer $m \geq 1$ such that $X_m \geq 1 - \varepsilon$.

(3) **[2 Points]** Show that $\max(X_1, \ldots, X_n)$ converges almost surely to 1 as $n \to \infty$.

(4) **[3 Points]** Is it true that $\max(X_1, \ldots, X_n)$ converges in L^1 to 1 as $n \to \infty$? Justify your answer.

Question 2

[**9 Points**] Let E be a non-empty set. Let $\mathcal{C}$ be a non-empty collection of subsets of E. We say that $\mathcal{C}$ is a monotone class if the following two properties are satisfied:
(i) for every sequence $(A_i)_{i\geq 1}$ of elements of $\mathcal{C}$ such that $A_i \subset A_{i+1}$ for every $i \geq 1$ we have $\bigcup_{i=1}^{\infty} A_i \in \mathcal{C}$;
(ii) for every sequence $(A_i)_{i\geq 1}$ of elements of $\mathcal{C}$ such that $A_{i+1} \subset A_i$ for every $i \geq 1$ we have $\bigcap_{i=1}^{\infty} A_i \in \mathcal{C}$.

(1) [**3 Points**] Assume that $\mathcal{C}$ is a monotone class which is stable by complementation (that is, if $A \in \mathcal{C}$, then $A^c \in \mathcal{C}$) and stable by finite unions (that is, for every $n \geq 1$, if $A_1, \ldots, A_n \in \mathcal{C}$, then $A_1 \cup A_2 \cup \cdots \cup A_n \in \mathcal{C}$). Show that $\mathcal{C}$ is a σ-field.

Let $\mathcal{D}$ be a non-empty collection of subsets of E.

(2) [**2 Points**] Assume that $\mathcal{D}$ is a Dynkin system. Show that for $A, B \in \mathcal{D}$ such that $A \subset B$ we have $B \backslash A \in \mathcal{D}$.

Hint. Write $B \backslash A = (B^c \cup A)^c$.

(3) [**3 Points**] Assume that $\mathcal{D}$ is a Dynkin system. Show that $\mathcal{D}$ is a monotone class.

(4) [**1 Point**] Give an example of a monotone class which is not a Dynkin system.

Question 3

[**13 Points**] Let $(X_i)_{i\geq 1}$ be a sequence of i.i.d. random variables following the uniform distribution on $[0,1]$.

(1) [**5 Points**] Show that $n\min(X_1,\ldots,X_n)$ converges in distribution to a random variable Z when $n\to\infty$ and give the law of Z.

(2) [**4 Points**] Show that

$$(X_1+\cdots+X_n)\min(X_1,\ldots,X_n) \quad \xrightarrow[n\to\infty]{(d)} \quad Z/2.$$

(3) [**1 Point**] Compute the variance of X_1.

(4) [**3 Points**] Show that the sequence of random variables

$$\frac{1}{\sqrt{n}}\cdot\frac{X_1+\cdots+X_n}{2(X_1+\cdots+X_n)-n}$$

converges in distribution as $n\to\infty$.

Question 4

[**14 Points**]
Set $X_0 = 0$ and let $(X_n)_{n\geq 1}$ be i.i.d. real-valued standard Gaussian $\mathcal{N}(0,1)$ random variables. For $i \geq 1$, we set

$$Y_i = \frac{X_i - X_{i-1}}{i}.$$

For $n \geq 0$, we define $\mathcal{F}_n = \sigma(X_i, i \leq n)$, as well as

$$S_n = \sum_{i=1}^{n} Y_i, \qquad T_n = S_n - \frac{X_n}{n+1}.$$

(1) [**1 Point**] Find a sequence $(u_i)_{i\geq 1}$ of positive real numbers such that for every $n \geq 0$ we have

$$T_n = \sum_{i=1}^{n} u_i X_i.$$

(2) [**4 Points**] Show that $(T_n)_{n\geq 0}$ is a $(\mathcal{F}_n)_{n\geq 0}$ martingale.

(3) [**2 Points**] Show that $(T_n)_{n\geq 0}$ is bounded in L^2.

(4) [**3 Points**] Show that $(T_n)_{n\geq 0}$ converges almost surely, in L^1 and in L^2.

(5) [**1 Point**] Show that for every $x > 0$ we have $\mathbb{P}(|X_1| \geq x) \leq \frac{2}{x\sqrt{2\pi}} e^{-x^2/2}$.

Hint. Use $\int_x^\infty e^{-t^2/2} \mathrm{dt} \leq \int_x^\infty \frac{t}{x} e^{-t^2/2} \mathrm{dt}$.

(6) [**3 Points**] Show that $(S_n)_{n\geq 0}$ converges almost surely, in L^1 and in L^2.

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