PROBABILITY THEORY (D-MATH)
EXERCISE SHEET 1

Exercise 1. This exercise shows that a simple random walk on $\mathbb{Z}$ cannot be confined to a strip forever. Let $(X_n)_{n\geq 1}$ be an iid sequence of random variables defined by

$$\mathrm{P}(X_1 = 1) = \mathrm{P}(X_1 = -1) = 1/2.$$

For $n \geq 1$, define $S_n = X_1 + \cdots + X_n$. Let $k \geq 1$ be a fixed integer. Show that

$$\mathrm{P}\big(\forall n \geq 1 \;\; 0 \leq S_n \leq k\big) = 0.$$

Exercise 2 [R]. Let $(X_n)_{n\geq 1}$ be an iid sequence of random variables uniformly distributed in $\{-1, 1, 2, -2\}$. For $n \geq 1$, let $S_n = X_1 + \cdots + X_n$. Fix $n \geq 1$.

(i) Compute $\mathrm{E}(S_n)$ and $\mathrm{Var}(S_n)$.

(ii) Prove that

$$\mathrm{P}(|S_n| \geq 2\sqrt{n}) \leq \frac{3}{4}.$$

(iii) Prove that

$$\forall k \in \mathbb{Z} \quad \mathrm{P}(S_n = k) = \mathrm{P}(S_n = -k).$$

(iv) Provethat

$$\forall k \in \mathbb{Z} \quad \mathrm{P}(X_1 + \cdots + X_n = k) = \mathrm{P}(X_{n+1} + \cdots + X_{2n} = k).$$

(v) Deduce that

$$\forall k \in \mathbb{Z} \quad \mathrm{P}(S_{2n} = k) = \sum_{i \in \mathbb{Z}} \mathrm{P}(S_n = i) \cdot \mathrm{P}(S_n = k - i).$$

(vi) Apply the Cauchy-Schwarz inequality to show that

$$\forall k \in \mathbb{Z} \quad \mathrm{P}(S_{2n} = k) \leq \mathrm{P}(S_{2n} = 0).$$

(vii) Deduce that

$$\mathrm{P}(S_{2n} = 0) \geq \frac{1}{50\sqrt{n}}.$$

Exercise 3. Let $(X_n)_{n\geq 1}$ be iid Exp(1) random variables. Show that

$$\limsup_{n\to\infty} \frac{X_n}{\log n} = 1 \quad a.s.$$

Exercise 4. Let $(A_n)_{n\geq 1}$ be a sequence of events such that

$$\lim_{n\to\infty} \mathrm{P}(A_n) = 0 \text{ and } \sum_{n\geq 1} \mathrm{P}(A_n \setminus A_{n+1}) < \infty.$$

Prove that P(infinitely many A_n occur) $= 0$.

Submission of solutions. Hand in your solutions by 18:00, 27/09/2024 following the instructions on the course website

`https://metaphor.ethz.ch/x/2024/hs/401-3601-00L/`

Note that only the exercises marked with [R] will be corrected.