PROBABILITY THEORY (D-MATH)
EXERCISE SHEET 1 – SOLUTION

Exercise 1. This exercise shows that a simple random walk on $\mathbb{Z}$ cannot be confined to a strip forever. Let $(X_n)_{n\geq 1}$ be an iid sequence of random variables defined by

$$\mathrm{P}(X_1 = 1) = \mathrm{P}(X_1 = -1) = 1/2.$$

For $n \geq 1$, define $S_n = X_1 + \cdots + X_n$. Let $k \geq 1$ be a fixed integer. Show that

$$\mathrm{P}\big(\forall n \geq 1 \;\; 0 \leq S_n \leq k\big) = 0.$$

Solution. For each $i \geq 1$ define

$$A_i = \{X_{2ik+1} = X_{2nk+2} = \cdots = X_{2ik+k+1} = 1\}.$$

We claim that for all $i \geq 1$,

$$A_i \subset \big\{\forall n \geq 1 \;\; 0 \leq S_n \leq k\big\}^c.$$

To check this, let $\omega \in A_i$. We show that then $\omega \in \big\{\forall n \geq 1 \;\; 0 \leq S_n \leq k\big\}^c$. If $S_{2ki}(\omega) < 0$, then we are done, and if $S_{2ki}(\omega) \geq 0$, then

$$S_{2ki+k+1} = S_{2ki} + X_{2ik+1} + \cdots + X_{2ik+k+1} \geq k + 1,$$

which proves the claim. So we have $\big\{\forall n \geq 1 \;\; 0 \leq S_n \leq k\big\} \subset A_i^c$, and so

$$\big\{\forall n \geq 1 \;\; 0 \leq S_n \leq k\big\} \subset \bigcap_{i\geq 1} A_i^c.$$

We show that $\mathrm{P}(\cap_{i\geq 1} A_i) = 0$, which would complete the proof. To see this, first note that by independence of (X_n) we have

$$\forall i \geq 1 \quad \mathrm{P}(A_i) = (1/2)^{k+1}.$$

Next, observe that, by construction, $(A_i)_{i\geq 1}$ are independent events, so for all $n \geq 1$ we have

$$\mathrm{P}\left(\bigcap_{i=1}^{n} A_i^c\right) = (1 - (1/2)^{k+1})^n.$$

Therefore,

$$\mathrm{P}\left(\bigcap_{i\geq 1} A_i^c\right) = \lim_{n\to\infty} \mathrm{P}\left(\bigcap_{i=1}^{n} A_i^c\right) = 0,$$

as desired.

Exercise 2 [R]. Let $(X_n)_{n\geq 1}$ be an iid sequence of random variables uniformly distributed in $\{-1, 1, 2, -2\}$. For $n \geq 1$, let $S_n = X_1 + \cdots + X_n$. Fix $n \geq 1$.

Solution.

(i) Compute $\mathrm{E}(S_n)$ and $\mathrm{Var}(S_n)$.

(ii) Prove that

$$\mathrm{P}(|S_n| \geq 2\sqrt{n}) \leq \frac{3}{4}.$$

(iii) Prove that

$$\forall k \in \mathbb{Z} \quad \mathrm{P}(S_n = k) = \mathrm{P}(S_n = -k).$$

(iv) Provethat

$$\forall k \in \mathbb{Z} \quad \mathrm{P}(X_1 + \cdots + X_n = k) = \mathrm{P}(X_{n+1} + \cdots + X_{2n} = k).$$

(v) Deduce that

$$\forall k \in \mathbb{Z} \quad \mathrm{P}(S_{2n} = k) = \sum_{i\in\mathbb{Z}} \mathrm{P}(S_n = i) \cdot \mathrm{P}(S_n = k - i).$$

(vi) Apply the Cauchy-Schwarz inequality to show that

$$\forall k \in \mathbb{Z} \quad \mathrm{P}(S_{2n} = k) \leq \mathrm{P}(S_{2n} = 0).$$

(vii) Deduce that

$$\mathrm{P}(S_{2n} = 0) \geq \frac{1}{50\sqrt{n}}.$$

Solution.

(i) Since X_1 is symmetric, $\mathrm{E}(X_1) = 0$. So, by linearity, we have

$$\mathrm{E}(S_n) = \mathrm{E}(X_1) + \cdots + \mathrm{E}(X_n) = 0.$$

We compute $\mathrm{Var}(X_1) = \mathrm{E}(X_1)^2 - \mathrm{E}(X_1)^2 = 5/2$. Since (X_n) are independent, we have

$$\mathrm{Var}(S_n) = \mathrm{Var}(X_1) + \cdot + \mathrm{Var}(X_n) = 5n/2.$$

(ii) By Chebyshev's inequality, we have

$$\mathrm{P}(|S_n| \geq 2\sqrt{n}) \leq \frac{(5n/2)}{4n} \leq 3/4.$$

(iii) Define a sequence of random variables $(Y_i)_{i\geq 1}$ by $Y_i = -X_i$. Observe that $(Y_i)_{n\geq 1}$ has the same distribution as $(X_i)_{i\geq 1}$. So

$$\forall k \in \mathbb{Z} \quad \mathrm{P}(X_1 + \cdots + X_n = k) = \mathrm{P}(Y_1 + \cdots + Y_n = k) = \mathrm{P}(X_1 + \cdots + X_n = -k)$$

(iv) This time define a sequence of random variables $(Y_i)_{i\geq 1}$ by $Y_i = X_{n+i}$. Again, $(Y_i)_{n\geq 1}$ has the same distribution as $(X_i)_{i\geq 1}$, so

$$\forall k \in \mathbb{Z} \quad \mathrm{P}(X_1 + \cdots + X_n = k) = \mathrm{P}(Y_1 + \cdots + Y_n = k) = \mathrm{P}(X_{n+1} + \cdots + X_{2n} = k).$$

(v) Fix $k \in \mathbb{Z}$. We have

$$
\begin{aligned}
\mathrm{P}(S_{2n} = k) &= \mathrm{P}\Bigg(\bigcup_{i\in\mathbb{Z}}\{S_n = i, S_{2n} - S_n = k - i\}\Bigg) \\
\text{(the union is disjoint)} \quad &= \sum_{i\in\mathbb{Z}} \mathrm{P}(S_n = i, S_{2n} - S_n = k - i) \\
(S_n \text{ and } S_{2n} - S_n \text{ are ind.}) \quad &= \sum_{i\in\mathbb{Z}} \mathrm{P}(S_n = i) \cdot \mathrm{P}(X_{n+1} + \cdots + X_{2n} = k - i) \\
\text{(part iv)} \quad &= \sum_{i\in\mathbb{Z}} \mathrm{P}(S_n = i) \cdot \mathrm{P}(S_n = k - i).
\end{aligned}
$$

(vi) We have

$$
\begin{aligned}
\mathrm{P}(S_{2n} = k) &= \sum_{i\in\mathbb{Z}} \mathrm{P}(S_n = i) \cdot \mathrm{P}(S_n = k - i) \quad \text{(part v)} \\
\text{(Cauchy-Schwarz)} \quad &\leq \Bigg(\sum_{i\in\mathbb{Z}} \mathrm{P}(S_n = i)^2\Bigg)^{1/2} \Bigg(\sum_{i\in\mathbb{Z}} \mathrm{P}(S_n = k - i)^2\Bigg)^{1/2} \\
(\mathbb{Z} = k - \mathbb{Z}) \quad &= \Bigg(\sum_{i\in\mathbb{Z}} \mathrm{P}(S_n = i)^2\Bigg)^{1/2} \Bigg(\sum_{i\in\mathbb{Z}} \mathrm{P}(S_n = i)^2\Bigg)^{1/2} \\
&= \sum_{i\in\mathbb{Z}} \mathrm{P}(S_n = i)^2 \\
\text{(part iii)} \quad &= \sum_{i\in\mathbb{Z}} \mathrm{P}(S_n = i)\mathrm{P}(S_n = -i) \\
\text{(part v)} \quad &= \mathrm{P}(S_{2n} = 0).
\end{aligned}
$$

(vii) Using part (ii) we have $\mathrm{P}\big(|S_{2n}| \in [-2\sqrt{2n}, 2\sqrt{2n}]\big) = 1 - \mathrm{P}(|S_n| > 2\sqrt{n}) \geq 1/4$. Let

$$
I = [-2\sqrt{2n}, 2\sqrt{2n}] \cap \mathbb{Z}.
$$

Then we have

$$
\begin{aligned}
1/4 &\leq \mathrm{P}(S_n \in I) \\
&= \mathrm{P}\Bigg(\bigcup_{i\in I} S_n = i\Bigg) \\
\text{(the union is disjoint)} \quad &= \sum_{i\in I} \mathrm{P}(S_n = i) \\
\text{(part vi)} \quad &\leq |I| \cdot \mathrm{P}(S_{2n} = 0) \\
\big(|I| \leq 10\sqrt{n}\big) \quad &\leq 10\sqrt{n} \cdot \mathrm{P}(S_{2n} = 0).
\end{aligned}
$$

So

$$
\mathrm{P}(S_{2n} = 0) \geq \frac{1}{40\sqrt{n}} > \frac{1}{50\sqrt{n}},
$$

which completes the proof.

Exercise 3. Let $(X_n)_{n\geq 1}$ be iid Exp(1) random variables. Show that

$$\limsup_{n\to\infty} \frac{X_n}{\log n} = 1 \quad a.s.$$

Solution. Let $\epsilon > 0$ be arbitrary. For $n \geq 1$, define the events

$$A_n = \{X_n \geq (1+\epsilon)\log n\} \quad \text{and} \quad B_n = \{X_n \geq (1-\epsilon)\log n\}.$$

First, note that

$$\sum_{n\geq 1} \mathrm{P}(A_n) = \sum_{n\geq 1} 1/n^{1+\epsilon} < \infty,$$

so by the first Borel-Cantelli lemma, only finitely many of the A_n's occur almost surely. Second, we have that

$$\sum_{n\geq 1} \mathrm{P}(B_n) = \sum_{n\geq 1} 1/n^{1-\epsilon} = \infty.$$

Since the B_n's are independent, the second Borel-Cantelli lemma implies that infintely many of the B_n's occur. So, we get

$$\limsup_{n\to\infty} \frac{X_n}{\log n} \in [1-\epsilon, 1+\epsilon] \quad a.s.$$

Now, we take a countable sequence $\epsilon_k \to 0$ to complete the proof. More precisely, we get for all $k \geq 1$,

$$\mathrm{P}\left(\limsup_{n\to\infty} \frac{X_n}{\log n} \in [1-1/k, 1+1/k] \right) = 1,$$

which implies

$$\mathrm{P}\left(\bigcap_{k\geq 1} \limsup_{n\to\infty} \frac{X_n}{\log n} \in [1-1/k, 1+1/k] \right) = 1.$$

This completes the proof because event in the display above is the same as

$$\{\limsup_{n\to\infty} X_n/\log n = 1\}.$$

Exercise 4. Let $(A_n)_{n\geq 1}$ be a sequence of events such that

$$\lim_{n\to\infty} \mathrm{P}(A_n) = 0 \text{ and } \sum_{n\geq 1} \mathrm{P}(A_n \setminus A_{n+1}) < \infty.$$

Prove that P(infinitely many A_n occur) $= 0$.

Solution.

We want to show that

$$\mathrm{P}\left(\bigcap_{n\geq 1}\bigcup_{m\geq n} A_m\right) = \lim_{n\to\infty} \mathrm{P}\left(\bigcup_{m\geq n} A_m\right) = 0.$$

We claim that

$$\bigcup_{m\geq n} A_m \subset \left(\bigcup_{m\geq n} A_m \setminus A_{m+1}\right) \cup \left(\bigcup_{m\geq n}\bigcap_{k\geq m} A_k\right). \tag{1}$$

To show this, let $\omega \in \cup_{m\geq n} A_m$ but $\omega \notin \cap_{m\geq n} A_m \setminus A_{m+1}$. Let $m' \geq n$ be such that $\omega \in A_{m'}$. Since $\omega \notin A_{m'} \setminus A_{m'+1}$, we must have $\omega \in A_{m'+1}$. Continuing like this, we get inductively that $\omega \in A_k$ for all $k \geq m'$. This shows $\omega \in \cap_{k\geq m'} A_k$, which proves the claim. Next, observe that for each $m \geq n$

$$\mathrm{P}\left(\bigcap_{k\geq m} A_k\right) = 0.$$

since $\bigcap_{k\geq n} A_k$ is contained in each of the $(A_k)_{k\geq m}$ and $\mathrm{P}(A_k) \to 0$. So

$$\mathrm{P}\left(\bigcup_{m\geq n}\bigcap_{k\geq m} A_k\right) = 0,$$

as well. Using this and union bound in (1) we obtain

$$\mathrm{P}\left(\bigcup_{m\geq n} A_m\right) \leq \sum_{k\geq n} \mathrm{P}(A_k \setminus A_{k+1}),$$

which converges to 0 as $n \to \infty$ since $\sum_{n\geq 1} \mathrm{P}(A_n \setminus A_{n+1}) < \infty$ by assumption. This completes the proof.