Prof. Vincent Tassion HS 2024

PROBABILITY THEORY (D-MATH)
EXERCISE SHEET 10

Exercise 1. [R] Let A be an compact set in $\mathbb{R}^2$ and let $(X, Y) \sim \mathrm{Unif}(A)$. Compute

$$\mathrm{E}\big(X^2|Y\big)$$

in the following cases:
(1) $A = [-1, 1]^2$,
(2) $A = \{(x, y) : |x| + |y| \le 1\}$.

Exercise 2. Let X, Y be independent random variables and let $\psi : \mathbb{R}^2 \to \mathbb{R}_{\ge 0}$ be a measurable function such that

$$\mathrm{E}\big(|\psi(X, Y)|\big) < \infty.$$

Define $\phi : \mathbb{R} \to [0, \infty]$ by

$$\phi(y) = \mathrm{E}\big(\psi(X, y)\big).$$

Show that

$$\mathrm{E}(\psi(X, Y)|Y) = \phi(Y) \ a.s.$$

Exercise 3. [R] Let $(Y_n)_{n\ge1}$ be iid random variables which are uniform in $\{-1, +1\}$ and let X be a random variable in L^2. Let $[n]$ denote $\{1, \dots, n\}$ and for a subset $S \subset [n]$, define

$$Y_S = \prod_{i\in S} Y_i,$$

where $Y_\emptyset$ defined to be 1.
(1) Show that $\mathrm{E}(X|Y_1) = \mathrm{E}(X) + \mathrm{E}(XY_1)Y_1$.
(2) More generally, for all $n \ge 1$ show that

$$\mathrm{E}(X|Y_1, \dots, Y_n) = \sum_{S\subset[n]} \mathrm{E}(XY_S)Y_S.$$

Exercise 4. [R] Let X be a real-valued random variable defined on $(\Omega, \mathcal{F}, \mathrm{P})$ that takes values in $[0, \infty]$ a.s. Let $\mathcal{G} \subset \mathcal{F}$ be a sigma-algebra. Define $\mathrm{E}(X|\mathcal{G})$ and show that it is unique (up to almost sure equivalence).

Submission of solutions. Hand in your solutions by 18:00, 29/11/2024 following the instructions on the course website

`https://metaphor.ethz.ch/x/2024/hs/401-3601-00L/`

Note that only the exercises marked with [R] will be corrected.