PROBABILITY THEORY (D-MATH)
EXERCISE SHEET 10 – SOLUTION

Exercise 1. [R] Let A be an compact set in $\mathbb{R}^2$ and let $(X, Y) \sim \mathrm{Unif}(A)$. Compute

$$\mathrm{E}\big(X^2|Y\big)$$

in the following cases:
(1) $A = [-1, 1]^2$,
(2) $A = \{(x, y) : |x| + |y| \leq 1\}$.

Solution.
(1) In this case, X and Y are independent $\mathrm{Unif}[-1, 1]$ random variables. So

$$\mathrm{E}(X^2|Y) = \mathrm{E}(X^2) = \int_{-1}^{1} x^2/2 \, dx = 1/3.$$

(2) In this case, (X, Y) has density given by

$$f_{(X,Y)}(x, y) = \frac{1_{|x|+|y|\leq 1}}{2}.$$

Therefore, the marginal density of y is given by

$$f_Y(y) = 1_{|y|\leq 1} \int_{|y|-1}^{1-|y|} 1/2 \, dy = (1 - |y|)1_{|y|\leq 1}.$$

We compute the conditional expectation as in section 5 of chapter 10. Let $y \in [-1, 1]$. We get

$$\begin{aligned}\phi(y) &= \int_{-1}^{1} \frac{x^2 1_{|x|+|y|\leq 1}}{2(1 - |y|)} \, dx \\ &= \frac{1}{2(1 - |y|)} \int_{|y|-1}^{1-|y|} x^2 \, dx \\ &= \frac{(1 - |y|)^2}{3}.\end{aligned}$$

So

$$\mathrm{E}(X^2|Y) = \frac{(1 - |Y|)^2}{3} \quad a.s.$$

Exercise 2. Let X, Y be independent random variables and let $\psi : \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ be a measurable function such that

$$\mathrm{E}\big(|\psi(X,Y)|\big) < \infty.$$

Define $\phi : \mathbb{R} \to [0, \infty]$ by

$$\phi(y) = \mathrm{E}\big(\psi(X, y)\big).$$

Show that

$$\mathrm{E}(\psi(X,Y)|Y) = \phi(Y) \ a.s.$$

Solution. First, since $\phi(y) = \int_{\mathbb{R}} \psi(x,y)\mu_X(dx)$, ϕ is a measurable function (this is part of Fubini's theorem). Hence, $h(Y)$ is $\sigma(Y)$-measurable. Therefore, it suffices to check that for every non-negative random variable Z which is $\sigma(Y)$-measurable, we have

$$\mathrm{E}\big(\psi(X,Y)Z\big) = \mathrm{E}\big(\phi(Y)Z\big).$$

To this end, using Fubini's theorem we get

$$\begin{aligned}
\mathrm{E}\big(\psi(X,Y)Z\big) &= \int_{\mathbb{R}\times\mathbb{R}\times\mathbb{R}_{\geq 0}} \psi(x,y) z \mu_{(X,Y,Z)}(dx dy dz) \\
\text{(because } X \text{ and } (Y,Z) \text{ are independent)} \quad &= \int_{\mathbb{R}\times\mathbb{R}\times\mathbb{R}_{\geq 0}} \psi(x,y) z \mu_X(dx)\mu_{(Y,Z)}(dy dz) \\
&= \int_{\mathbb{R}\times\mathbb{R}_{\geq 0}} z\bigg(\int_{\mathbb{R}} \psi(x,y)\mu_X(dx)\bigg)\mu_{(Y,Z)}(dy dz) \\
&= \int_{\mathbb{R}\times\mathbb{R}_{\geq 0}} z\phi(y)\mu_{(Y,Z)}(dy dz) \\
&= \mathrm{E}(\phi(Y)Z).
\end{aligned}$$

Exercise 3. [R] Let $(Y_n)_{n\geq 1}$ be iid random variables which are uniform in $\{-1,+1\}$ and let X be a random variable in L^2. Let $[n]$ denote $\{1,\dots,n\}$ and for a subset $S\subset[n]$, define

$$Y_S=\prod_{i\in S}Y_i,$$

where $Y_\emptyset$ defined to be 1.
(1) Show that $\mathrm{E}(X|Y_1)=\mathrm{E}(X)+\mathrm{E}(XY_1)Y_1$.
(2) More generally, for all $n\geq 1$ show that

$$\mathrm{E}(X|Y_1,\dots,Y_n)=\sum_{S\subset[n]}\mathrm{E}(XY_S)Y_S.$$

Solution. We use the notation $Y=(Y_1,\dots,Y_n)$. We prove the general formula using the L^2 projection interpretation of conditional expectation, which is applicable because we assume that $X\in L^2$. Let

$$\mathcal{F}=\sigma(Y_1,\dots,Y_n)$$

and let $\mathcal{H}_\mathcal{F}$ be the vector space of $\mathcal{F}$-measurable random variables in L^2. We first establish that $\mathcal{H}_\mathcal{F}$ has dimension at most 2^n. Consider the random variables

$$\{y\in\{-1,1\}^n:1_{Y=y}\}.$$

Being a functions of Y and bounded, these random variables are in $H_\mathcal{G}$. We show that they span $\mathcal{H}_\mathcal{F}$. Let Z be a $\mathcal{F}$-measurable random variable in L^2. Then we know that there exists a measurable function $f:\mathbb{R}^n\to\mathbb{R}$ such that

$$Z=f(Y_1,\dots,Y_n).$$

But now

$$f(Y_1,\dots,Y_n)=\sum_{y\in\{-1,1\}^n}f(y)1_{Y=y}.$$

This shows that the dimension of $H_\mathcal{F}$ is at most 2^n. Next, we claim that

$$\{S\subset[n]:Y_S\}$$

is an orthonormal basis of $H_\mathcal{F}$. First, since $Y_S\in\{-1,1\}$ we have $\mathrm{E}(Y_S^2)=1$. Next let R,S be distinct subsets of $[n]$. Then

$$\mathrm{E}(Y_RY_S)=\mathrm{E}(Y_{R\Delta S})=\mathrm{E}\Bigg(\prod_{k\in R\Delta S}Y_k\Bigg)=\prod_{k\in R\Delta S}\mathrm{E}(Y_k)=0,$$

where we used independence in the second last inequality, and the facts that $R\Delta S\neq\emptyset$ and $\mathrm{E}(Y_k)=0$ for all k in the last. This shows orthonormality. Since we have 2^n such functions, they form a basis. So finally, the formula we wanted to show is just the projection formula with respect to this orthonormal basis.

Exercise 4. [R] Let X be a real-valued random variable defined on $(\Omega, \mathcal{F}, \mathrm{P})$ that takes values in $[0, \infty]$ a.s. Let $\mathcal{G} \subset \mathcal{F}$ be a sigma-algebra. Define $\mathrm{E}(X|\mathcal{G})$ and show that it is unique (up to almost sure equivalence).

Solution. Refer to section 7 of chapter 10.