PROBABILITY THEORY (D-MATH)
EXERCISE SHEET 11

Exercise 1. [R] Let $(X_n)_{n\geq 1}$ be iid random variables in L^1 and for $n \geq 1$, let

$$S_n = X_1 + \cdots + X_n.$$

Compute $\mathrm{E}(S_n|X_1)$ and $\mathrm{E}(X_1|S_n)$.

Exercise 2. [R] Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a probability space. Let $\mathcal{G}, \mathcal{H} \subset \mathcal{F}$ be sigma-algebras and let X be a random variable. Show that we need not have that

$$\mathrm{E}(\mathrm{E}(X|\mathcal{G})|\mathcal{H}) = \mathrm{E}(X|\mathcal{G} \cap \mathcal{H}).$$

Exercise 3. [R] Let $(X_n)_{n\geq 1}$ be iid random variables taking values in $\{+1. -1\}$ with $\mathrm{P}(X_1 = 1) = 1/2$. Let $S_0 = 0$ and for $n \geq 1$, let $S_n = X_1 + \cdots + X_n$. Let $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and for $n \geq 1$, let $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$. Show that

$$M_n = S_n^2 - n$$

is a $(\mathcal{F}_n)$-martingale.

Exercise 4. Fix $p \in (0, 1)$. Let $(X_n)_{n\geq 1}$ be iid random variables taking values in $\{+1, -1\}$ with $\mathrm{P}(X_1 = 1) = p$. Let $S_0 = 0$ and for $n \geq 1$ let $S_n = X_1 + \cdots + X_n$. Let $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and for $n \geq 1$, let $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$. Show that

$$M_n = \left(\frac{1}{p} - 1\right)^{S_n}$$

is a $(\mathcal{F}_n)$-martingale.

Exercise 5 (Azuma's inequality). [R] Let $(X_n)_{n\geq 0}$ be martingale with respect to its canonical filtration $(\mathcal{F}_n)_{\geq 0}$. Assume $X_0 = 0$ and that $|X_n - X_{n-1}| \leq 1$ for all $n \geq 1$. Fix $m \geq 1$. The aim of this exercise is to show that $\lambda > 0$ we have

$$\mathrm{P}(X_m > \lambda\sqrt{m}) \leq e^{-\lambda^2/2}. \tag{1}$$

(1) Let $\alpha > 0$. Show that for all $x \in [-1, 1]$ we have $e^{\alpha x} \leq \frac{e^{\alpha}+e^{-\alpha}}{2} + \frac{e^{\alpha}-e^{-\alpha}}{2}x$
(2) Set $Y_i = X_i - X_{i-1}$. Show that for all $i \geq 1$ we have

$$\mathrm{E}(e^{\alpha Y_i}|\mathcal{F}_{i-1}) \leq \cosh(\alpha) \leq e^{\alpha^2/2}.$$

(3) Deduce that $\mathrm{E}(e^{\alpha X_m}) \leq e^{\alpha^2 m/2}$.
(4) Use $\alpha = \lambda/\sqrt{m}$ and Markov's inequality to prove (1).

Submission of solutions. Hand in your solutions by 18:00, 06/12/2024 following the instructions on the course website

`https://metaphor.ethz.ch/x/2024/hs/401-3601-00L/`

Note that only the exercises marked with [R] will be corrected.