PROBABILITY THEORY (D-MATH)
EXERCISE SHEET 12

Exercise 1. [R]
(1) Let $(X_n)_{n\geq 1}$ be an iid sequence of random variables uniform in $\{-1,1\}$. Show that
$$S_n = \sum_{m=1}^{n} \frac{X_m}{m^{3/4}}$$
converges almost surely as $n \to \infty$.
(2) Find an example of a martingale that converges almost surely but is not bounded in L^1.
(3) Find an example of a martingale that converges almost surely to ∞.

Exercise 2. Let $(Y_n)_{n\geq 0}$ be a sequence of non-negative iid random variables with $\mathrm{E}(Y_1) = 1$ and $\mathrm{P}(Y_1 = 1) < 1$ and let $(\mathcal{F}_n)_{n\geq 0}$ be the canonical filtration.
(1) Show that $X_n = \prod_{k=0}^{n} Y_k$ defines a martingale with respect to $(\mathcal{F}_n)$.
(2) Show that $X_n \to 0$ as $n \to \infty$ a.s.

Exercise 3. Fix $p \in (0, 1/2)$. Let $(X_n)_{n\geq 1}$ be iid random variables taking values in $\{-1,1\}$ with $\mathrm{P}(X_1 = 1) = p$. For $n \geq 1$ let $S_n = X_1 + \cdots + X_n$ and let
$$M_n = \left(\frac{1}{p} - 1\right)^{S_n}.$$
Show that M_n converges almost surely to 0 but $\mathrm{E}(M_n)$ does not converge to 0 as $n \to \infty$.

Exercise 4 (Positive harmonic functions on the square lattice). Let
$$h : \mathbb{Z}^2 \to \mathbb{R}_{>0}$$
be a harmonic function, meaning that
$$\forall (x,y) \in \mathbb{Z}^2 \quad h(x,y) = \frac{1}{4}\big(h(x+1,y) + h(x-1,y) + h(x,y+1) + h(x,y-1)\big).$$
The aim of this exercise is to show that h must be constant. Let $(X_n)_{n\geq 1}$ be iid uniform in $\{(1,0), (-1,0), (0,1), (0,-1)\}$. Define the sequence $(Z_n)_{n\geq 0}$ by $Z_0 = (0,0)$ and
$$Z_n = \sum_{k=1}^{n} X_k$$
for $n \geq 1$. Let $(\mathcal{F}_n)$ be the filtration generated by (Z_n).
(1) Show that $\big(h(Z_n)\big)_{n\geq 0}$ is a $\mathcal{F}_n$-martingale that converges almost surely.
(2) You may use the fact that
$$\forall (x,y) \in \mathbb{Z}^2 \quad |\{n : Z_n = (x,y)\}| = \infty \quad a.s.$$
Conclude that h is consant.
(3) Instead of assuming h takes positive values, assume that $|h|$ is bounded. Then show that h is constant.

Exercise 5 (Pólya's Urn). At time 0, an urn contains 1 black ball and 1 white ball. At each time $n \geq 1$ a ball is chosen at random from the urn and is replaced together with a new ball of the same colour. Just after time n, there are therefore $n+2$ balls in the urn, of which $B_n + 1$ are black, where B_n is the number of black balls chosen by time n. We let $\mathcal{F}_n = \sigma(B_1, \ldots, B_n)$.

(1) Prove that B_n is uniformly distributed on $\{0, 1, \ldots, n\}$.
(2) Let $M_n = (B_n + 1)/(n+2)$ be the proportion of black balls in the urn just after time n. Prove that (M_n) is a martingale with respect to $(\mathcal{F}_n)$ and show that $M_n \to U$ as $n \to \infty$ a.s. for some random variable U.
(3) Show that U is uniformly distributed on $(0, 1)$.

Submission of solutions. Hand in your solutions by 18:00, 13/12/2024 following the instructions on the course website

`https://metaphor.ethz.ch/x/2024/hs/401-3601-00L/`

Note that only the exercises marked with [R] will be corrected.