PROBABILITY THEORY (D-MATH)
EXERCISE SHEET 12 – SOLUTION

Exercise 1. [R]

(1) Let $(X_n)_{n\geq 1}$ be an iid sequence of random variables uniform in $\{-1,1\}$. Show that

$$S_n = \sum_{m=1}^{n} \frac{X_m}{m^{3/4}}$$

converges almost surely as $n \to \infty$.

(2) Find an example of a martingale that converges almost surely but is not bounded in L^1.

(3) Find an example of a martingale that converges almost surely to ∞.

Solution.

(1) Note that (S_n) is a martingale and that

$$\mathrm{E}(S_n^2) = \sum_{m=1}^{n} m^{-3/2} < C < \infty,$$

for some constant C not depending on n, showing that (S_n) is bounded in L^2. Therefore, S_n is also bounded in L^1, and therefore converges almost surely.

(2) Let $(X_n)_{n\geq 1}$ be independent random variables with law given by

$$\mathrm{P}(X_n = 4^n) = \mathrm{P}(X_n = -4^n) = \frac{1}{2^n}, \qquad \mathrm{P}(X_n = 0) = 1 - \frac{1}{2^{n-1}}$$

and $M_n = X_1 + \cdots + X_n$.

Since $\mathrm{E}X_n = 0$ for every $n \geq 1$, (M_n) is a martingale with respect to its canonical filtration. Since $\sum_{n\geq 1} \frac{1}{2^{n-1}} < \infty$, Borel-Cantelli 1 implies that $\mathrm{P}(\limsup\{|X_n| = 4^n\}) = 0$. Thus with probability 0 we have $X_n = 4^n$ infinitely often. In other words, a.s. $X_n = 0$ for n sufficiently large, which implies that a.s. M_n converges.

But if $X_n = 4^n$, then $M_n \geq 4^n - 4^{n-1} - \cdots - 1 \geq 4^{n-1}$, so

$$\mathrm{E}(|M_n|) \geq \mathrm{E}(|M_n| 1_{X_n = 4^n}) \geq 4^{n-1} \Pr X_n = 4^n = 2^{n-2} \underset{n\to\infty}{\longrightarrow} \infty,$$

so (M_n) is not bounded in L^1.

(3) Let $(X_n)_{n\geq 1}$ be independent random variables with law given by

$$\mathrm{P}(X_n = 1) = \frac{n^2}{n^2+1}, \qquad \mathrm{P}(X_n = -n^2) = \frac{1}{n^2+1}$$

and $M_n = X_1 + \cdots + X_n$. Since $\mathrm{E}(X_n) = 0$ for every $n \geq 1$, (M_n) is a martingale with respect to its canonical filtration. Since $\sum_{n\geq 1} \frac{1}{n^2+1} < \infty$, by Borel-Cantelli 1 we have $\mathrm{P}(\limsup\{X_n = -n^2\}) = 0$. Thus with probability 0 we have $X_n = -n^2$ infinitely often. In other words a.s. $X_n = 1$ for every n sufficiently large, so $M_n \to \infty$ a.s.

Exercise 2. Let $(Y_n)_{n\geq 0}$ be a sequence of non-negative iid random variables with $\mathrm{E}(Y_1) = 1$ and $\mathrm{P}(Y_1 = 1) < 1$ and let $(\mathcal{F}_n)_{n\geq 0}$ be the canonical filtration.

(1) Show that $X_n = \prod_{k=0}^n Y_k$ defines a martingale with respect to $(\mathcal{F}_n)$.
(2) Show that $X_n \to 0$ as $n \to \infty$ a.s.

Solution.

(1) Clearly X_n is $\mathcal{F}_n$-measurable. In addition, $X_n \geq 0$ and $\mathrm{E}(X_n) = \prod_{i=1}^n \mathrm{E}(Y_i) = 1$ for all $n \geq 1$. Thus $X_n \in L^1(\Omega, \mathcal{F}_n, \mathrm{P})$. Also $\mathrm{E}(X_{n+1} \mid \mathcal{F}_n) = \prod_{i=1}^n Y_i \cdot \mathrm{E}(Y_{n+1}) = X_n$, which implies that (X_n) is a $(\mathcal{F}_n)$ martingale.

(2) If $\mathrm{P}(Y_1 = 0) > 0$, then since the events $(\{Y_i = 0\})_{i\geq 1}$ are independent, by the second Borel-Cantelli Lemma, a.s. $\{Y_i = 0\}$ happens infinitely many times. This implies that $X_n = 0$ for all n large enough a.s.

Let us now suppose then that $Y_1 > 0$ almost surely. We show that $\ln(X_n) \to -\infty$ as $n \to \infty$ a.s., which will imply the desired result.

First case. $\ln(Y_1)$ is integrable. Then by using the strict concavity of the logarithm we get $\mathrm{E}(\ln Y_1) < \ln \mathrm{E}(Y_1) = 0$. Then by the strong law of large numbers

$$\frac{1}{n}\ln(X_n) = \frac{1}{n}\sum_{i=1}^n \ln(Y_i) \quad \underset{n\to\infty}{\longrightarrow} \quad \mathrm{E}(\ln(Y_1)) < 0.$$

almost surely. Thus $\ln(X_n) \to -\infty$ a.s.

Second case. $\ln(Y_1)$ is not integrable. Then by monotone convergence $\mathrm{E}(\ln\max(Y_1, \epsilon)) \to -\infty$ as $\varepsilon \to 0$, so we can choose $\varepsilon > 0$ such that $\mathrm{E}(\ln\max(Y_1, \epsilon)) < 0$. Then by the strong law of large numbers

$$\frac{1}{n}\ln(X_n) \leq \frac{1}{n}\sum_{i=1}^n \ln(\max(Y_i, \epsilon)) \quad \underset{n\to\infty}{\longrightarrow} \quad \mathrm{E}(\ln\max(Y_1 \vee \epsilon)) < 0$$

Thus $\ln(X_n) \to -\infty$ a.s.

Exercise 3. Fix $p \in (0, 1/2)$. Let $(X_n)_{n\geq 1}$ be iid random variables taking values in $\{-1, 1\}$ with $\mathrm{P}(X_1 = 1) = p$. For $n \geq 1$ let $S_n = X_1 + \cdots + X_n$ and let

$$M_n = \left(\frac{1}{p} - 1\right)^{S_n}.$$

Show that M_n converges almost surely to 0 but $\mathrm{E}(M_n)$ does not converge to 0 as $n \to \infty$.

Solution.

Note that $\mathrm{E}(X_1) < 0$ and $1/p - 1 > 1$ since $p \in (0, 1/2)$. By the strong the law of large numbers almost surely $S_n/n \to \mathrm{E}(X_1)$, and so $S_n \to -\infty$ almost surely. Therefore, $M_n \to 0$ almost surely. We saw in exercise 4 of sheet 11 that M_n is a martingale. So $\mathrm{E}(M_n) = \mathrm{E}(M_1) = 1$ for all n.

Exercise 4 (Positive harmonic functions on the square lattice). Let

$$h : \mathbb{Z}^2 \to \mathbb{R}_{>0}$$

be a harmonic function, meaning that

$$\forall (x,y) \in \mathbb{Z}^2 \quad h(x,y) = \frac{1}{4}\big(h(x+1,y) + h(x-1,y) + h(x,y+1) + h(x,y-1)\big).$$

The aim of this exercise is to show that h must be constant. Let $(X_n)_{n\geq 1}$ be iid uniform in $\{(1,0),(-1,0),(0,1),(0,-1)\}$. Define the sequence $(Z_n)_{n\geq 0}$ by $Z_0 = (0,0)$ and

$$Z_n = \sum_{k=1}^{n} X_k$$

for $n \geq 1$. Let $(\mathcal{F}_n)$ be the filtration generated by (Z_n).

(1) Show that $\big(h(Z_n)\big)_{n\geq 0}$ is a $\mathcal{F}_n$-martingale that converges almost surely.

(2) You may use the fact that

$$\forall (x,y) \in \mathbb{Z}^2 \quad |\{n : Z_n = (x,y)\}| = \infty \quad a.s.$$

Conclude that h is consant.

(3) Instead of assuming h takes positive values, assume that $|h|$ is bounded. Then show that h is constant.

Solution.

(1) $h(Z_n)$ is Z_n measurable so it is $\mathcal{F}_n$ measurable. Next, since Z_n takes only finitely many values, $h(Z_n)$ is bounded and hence in L^1. Now, we check the martingale property. Let $n \geq 0$. Let $z_1, \ldots, z_k$ be the possible values that Z_n can take. Then we have

$$\begin{aligned}
\mathrm{E}(h(Z_{n+1}|\mathcal{F}_n) &= \sum_{i=1}^{k} \mathrm{E}(h(Z_{n+1})1_{Z_n=z_i})|\mathcal{F}_n) \\
&= \sum_{i=1}^{k} \mathrm{E}(h(z_i + X_n)1_{Z_n=z_i})|\mathcal{F}_n) \\
(Z_n \text{ is } \mathcal{F}_n\text{-measurable}) \quad &= \sum_{i=1}^{k} 1_{Z_n=z_i}\mathrm{E}(h(z_i + X_{n+1}))|\mathcal{F}_n) \\
(X_{n+1} \text{ is independent from } \mathcal{F}_n) \quad &= \sum_{i=1}^{k} 1_{Z_n=z_i}\mathrm{E}(h(z_i + X_{n+1})) \\
(h \text{ is harmonic}) \quad &= \sum_{i=1}^{k} 1_{Z_n=z_i}h(z_i) \\
&= h(Z_n),
\end{aligned}$$

as required.

Finally, $h(Z_n)$ is a positive martingale so it is bounded in L^1 and so it converges almost surely.

(2) Let $\omega \in \Omega$ be such that the event in the question holds and such that $h(Z_n(\omega))$ converges to say l. (The set of such ω has probability 1 so in particular we can find one such.) Now, fix $z \in \mathbb{Z}^2$. Then there exists infinitely many n such that $Z_n = z$. Therefore, we must have $h(z) = l$. Therefore, h must be the constant function l.

(3) An analogous proof works using martingale convergence. Alternatively, one can shift the bounded harmonic function by a constant to ensure that it is positive and then apply the proof we just presented.

Exercise 5 (Pólya's Urn). At time 0, an urn contains 1 black ball and 1 white ball. At each time $n \geq 1$ a ball is chosen at random from the urn and is replaced together with a new ball of the same colour. Just after time n, there are therefore $n+2$ balls in the urn, of which $B_n + 1$ are black, where B_n is the number of black balls chosen by time n. We let $\mathcal{F}_n = \sigma(B_1, \ldots, B_n)$.

(1) Prove that B_n is uniformly distributed on $\{0, 1, \ldots, n\}$.
(2) Let $M_n = (B_n + 1)/(n+2)$ be the proportion of black balls in the urn just after time n. Prove that (M_n) is a martingale with respect to $(\mathcal{F}_n)$ and show that $M_n \to U$ as $n \to \infty$ a.s. for some random variable U.
(3) Show that U is uniformly distributed on $(0, 1)$.

Solution.

(1) We prove the claim by induction; when $n = 0$ the claim is obvious. Let us now consider the induction step. Then by the problem description,
for $b \in \{0, \ldots, n+1\}$, we have

$$\mathrm{E}(1_{B_{n+1}=b} \mid \mathcal{F}_n) = \frac{B_n + 1}{n+2} 1_{b=B_n+1} + \frac{n+1-B_n}{n+2} 1_{b=B_n}. \tag{1}$$

Thus by taking expectations on both sides and using the fact that B_n is uniform on $\{0, \ldots, n\}$ we get

$$\mathrm{P}(B_{n+1} = b) = \frac{b}{n+2} \cdot \frac{1}{n+1} + \frac{n+1-b}{n+2} \cdot \frac{1}{n+1} = \frac{1}{n+2}$$

as required.

(2) Since M_n is a function of B_n, it is $\mathcal{F}_n$ measurable. Also, $M_n \in L^1$ since it takes values in $[0, 1]$. By (1) we get

$$\begin{aligned} \mathrm{E}(M_{n+1} \mid \mathcal{F}_n) &= \sum_{b=0}^{n+2} \frac{b+1}{n+3} \mathrm{E}(1_{B_{n+1}=b} \mid \mathcal{F}_n) \\ &= \frac{B_n + 2}{n+3} \cdot \frac{B_n + 1}{n+2} + \frac{B_n + 1}{n+3} \cdot \frac{n+1-B_n}{n+2} = M_n \quad \text{a.s.} \end{aligned}$$

Since each M_n takes values in $[0, 1]$, (M_n) is bounded in L^1, so the martingale convergence theorem implies that the limit $U = \lim_{n\to\infty} M_n$ almost surely exists.

(3) Let $f : [0, 1] \to \mathbb{R}$ be continuous. We have

$$\mathrm{E}(f(M_n)) = \frac{1}{n+1} \sum_{i=1}^{n+1} f\left(\frac{i}{n+2}\right) \quad \underset{n\to\infty}{\longrightarrow} \quad \int_0^1 f(t)\mathrm{d}t, \tag{2}$$

by Riemann's theorem. This can be proved by hand: since f is continuous on $[0, 1]$ it is also uniformly continuous, so for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|x - y| \leq \delta$ implies $|f(x) - f(y)| \leq \varepsilon$. Then, for n such that $1/n < \delta$ we have $|\frac{i}{n+2} - \frac{i}{n+1}| \leq \frac{1}{n}$, so

$$\left| \frac{1}{n+1} \sum_{i=1}^{n+1} f\left(\frac{i}{n+2}\right) - \frac{1}{n+1} \sum_{i=1}^{n+1} f\left(\frac{i}{n+1}\right) \right| \leq \varepsilon$$

and also

$$\begin{aligned}\left|\frac{1}{n+1}\sum_{i=1}^{n+1} f\left(\frac{i}{n+1}\right) - \int_0^1 f(t)\mathrm{d}t\right| &= \left|\sum_{i=1}^{n+1}\int_{\frac{i-1}{n+1}}^{\frac{i}{n+1}}\left(f\left(\frac{i}{n+1}\right) - f(t)\mathrm{d}t\right)\right| \\ &\leq \sum_{i=1}^{n+1}\int_{\frac{i-1}{n+1}}^{\frac{i}{n+1}}\left|f\left(\frac{i}{n+1}\right) - f(t)\mathrm{d}t\right| \\ &\leq \varepsilon.\end{aligned}$$

which implies (2).

But $M_n \to U$ almost surely, so by continuity of f we also have $f(M_n) \to f(U)$ almost surely. Since f is continuous on $[0,1]$ is it bounded, so by dominated convergence we have $\mathrm{E}(f(M_n)) \to \mathrm{E}(f(U))$.

By (2) we conclude that

$$\int_{\mathbb{R}} f(x)\mu_U(\mathrm{d}x) = \mathrm{E}(f(U)) = \int_0^1 f(x)\mathrm{d}x.$$

It follows that U has the uniform distribution on $[0,1]$.