PROBABILITY THEORY (D-MATH)
EXERCISE SHEET 13

Exercise 1. Let $(\mathcal{F}_n)_{n\geq 0}$ be a filtration and let S, T be two stopping times with respect to $(\mathcal{F}_n)_{n\geq 0}$. Let $S, T : \Omega \to \mathbb{N} \cup \{\infty\}$ be $(\mathcal{F}_n)$ stopping times. Prove or disprove with a counter-example the following statements:

(1) $S \vee T$ is a stopping time.
(2) $S \wedge T$ is a stopping time.
(3) $S + T$ is a stopping time.
(4) $S + 1$ is a stopping time.
(5) $S - 1$ is a stopping time.

Exercise 2. [R] Let $(X_n)_{n\geq 1}$ be iid random variables uniform in $\{-1, 1\}$. Let $S_0 = 0$ and for $n \geq 1$ let $S_n = X_1 + \cdots + X_n$. Fix integers $a < 0 < b$. For an integer k, define $T_k = \min\{n \geq 0 : S_n = a\}$. Define

$$T_{a,b} = T_a \wedge T_b.$$

(1) Show that $T_{a,b}$ is a stopping time that is finite almost surely.
(2) Compute $\mathrm{P}(T_a < T_b)$.
(3) Compute $\mathrm{E}(T_{a,b})$.

Exercise 3. [R] Let $(M_n)_{n\geq 0}$ be a $(\mathcal{F}_n)_{n\geq 0}$ martingale and let T be a $(\mathcal{F}_n)_{n\geq 0}$ stopping time.

(1) Assume that $\mathrm{E}(T) < \infty$ and there there exists $K > 0$ such that a.s. we have

$$\mathrm{E}(|M_{n+1} - M_n|) \mid \mathcal{F}_n \leq K$$

for every $n \geq 0$. Show that $\mathrm{E}(M_T) = \mathrm{E}(M_0)$.
Hint. Justify that $|M_{T\wedge n}| \leq |M_0| + \sum_{i=0}^{\infty} |M_{i+1} - M_i| 1_{T>i}$ and use dominated convergence.

(2) Let $(X_n)_{n\geq 1}$ be iid L^1 real-valued random variables. Set $S_0 = 0$, $S_n = X_1 + \cdots + X_n$ for $n \geq 1$ and $\mathcal{F}_n = \sigma(S_i : 0 \leq i \leq n)$ for $n \geq 0$. Finally, let T be a $(\mathcal{F}_n)$-stopping time with $\mathrm{E}(T) < \infty$. Show that

$$\mathrm{E}(S_T) = \mathrm{E}(X_1)\mathrm{E}(T).$$

Exercise 4. Let $(M_n)_{n\geq 0}$ be a uniformly integrable martingale with respect to a filtration $(\mathcal{F}_n)_{n\geq 0}$.

(1) Is it true that the collection $\{M_T : T \text{ stopping time with respect to } (\mathcal{F}_n)_{n\geq 0}\}$ is uniformly integrable?
(2) Let T be a stopping time. Is it true that $(M_{n\wedge T})_{n\geq 0}$ is a uniformly integrable martingale? Justify your answer.

Submission of solutions. Hand in your solutions by 18:00, 20/12/2024 following the instructions on the course website

`https://metaphor.ethz.ch/x/2024/hs/401-3601-00L/`

Note that only the exercises marked with [R] will be corrected.