PROBABILITY THEORY (D-MATH)
EXERCISE SHEET 13 – SOLUTION

Exercise 1. Let $(\mathcal{F}_n)_{n\geq 0}$ be a filtration and let S, T be two stopping times with respect to $(\mathcal{F}_n)_{n\geq 0}$. Let $S, T : \Omega \to \mathbb{N} \cup \{\infty\}$ be $(\mathcal{F}_n)$ stopping times. Prove or disprove with a counter-example the following statements:

(1) $S \vee T$ is a stopping time.
(2) $S \wedge T$ is a stopping time.
(3) $S + T$ is a stopping time.
(4) $S + 1$ is a stopping time.
(5) $S - 1$ is a stopping time.

Solution.

(1) This is true. Indeed for $n \geq 0$ we have $\{S \vee T \leq n\} = \{S \leq n\} \cap \{T \leq n\} \in \mathcal{F}_n$ for $n \geq 0$ since $\{S \leq n\}, \{T \leq n\} \in \mathcal{F}_n$.

(2) This is true. For $n \geq 0$ we have $\{S \wedge T > n\} = \{S > n\} \cap \{T > n\}$ which is an element of $\mathcal{F}_n$ since $\mathcal{F}_n$ is stable under intersections and $\{S > n\} = \{S \leq n\}^c, \{T > n\} = \{T \leq n\}^c \in \mathcal{F}_n$. Therefore also $\{S \wedge T \leq n\} = \{S \wedge T > n\}^c \in \mathcal{F}_n$ as required.

(3) This is also true. Indeed, we have

$$\{S + T \leq n\} = \bigcup_{k+\ell \leq n} \{S \leq k\} \cap \{T \leq \ell\} \, .$$

Also $\{S \leq k\} \in \mathcal{F}_k \subset \mathcal{F}_n$ and $\{T \leq \ell\} \in \mathcal{F}_\ell \subset \mathcal{F}_n$ for $k, \ell \leq n$. Thus $\{S+T \leq n\} \in \mathcal{F}_n$ for all $n \geq 0$ as required.

(4) This is true. Indeed, for $n \geq 1$ we have $\{S + 1 \leq n\} = \{S \leq n - 1\} \in \mathcal{F}_{n-1} \subset \mathcal{F}_n$ and $\{S + 1 = 0\} = \emptyset$.

(5) This is not true. For instance, consider a Bernoulli random variable B with parameter $1/2$ and let $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_n = \sigma(B)$ for $n \geq 1$. Then $T := B + 1$ is an $(\mathcal{F}_n)$ stopping time but $\{T - 1 = 0\} = \{B = 0\} \notin \mathcal{F}_0$.

Exercise 2. [R] Let $(X_n)_{n\geq 1}$ be iid random variables uniform in $\{-1, 1\}$. Let $S_0 = 0$ and for $n \geq 1$ let $S_n = X_1 + \cdots + X_n$. Fix integers $a < 0 < b$. For an integer k, define $T_k = \min\{n \geq 0 : S_n = a\}$. Define

$$T_{a,b} = T_a \wedge T_b.$$

(1) Show that $T_{a,b}$ is a stopping time that is finite almost surely.
(2) Compute $\mathrm{P}(T_a < T_b)$.
(3) Compute $\mathrm{E}(T_{a,b})$.

Solution.

(1) We write $T = T_{a,b}$ for simplicity. We saw in the lectures that T_a and T_b are stopping times, so exercise 1 of this sheet shows that $T_{a,b}$ is also a stopping time. The proof that T is finite almost surely is similar to the one in exercise 1 of exercise sheet 1. We will also use the following observation. The proof that T is finite almost surely also gives the bound

$$\forall n \geq 1 \quad \mathrm{P}(T > n) \leq \delta^{n+o(n)},$$

for some $\delta \in (0, 1)$. This shows that $\mathrm{E}(T) < \infty$ as well.

(2) Note that (S_n) is a martingale. We verify that optional stopping holds for the stopping time T. Indeed, (1) shows $T < \infty$ a.s. and all random variables in the sequence $(S_{n\wedge T})$ lie in the interval $[b, a]$ (by the definition of T) so this sequence is uniformly integrable. By a result in the lecture, we have

$$\mathrm{E}(S_T) = 0.$$

But

$$\mathrm{E}(S_T) = \mathrm{P}(T_a < T_b)a + (1 - \mathrm{P}(T_a < T_b))b,$$

so we get

$$\mathrm{P}(T_a < T_b) = -b/(a - b).$$

(3) Consider the martingale $(M_n = S_n - n)$ (we verified this in exercise sheet 11). As in the previous part, to apply optional stopping for the stopping time T, we check that (M_n) is UI. For all $n \geq 1$ we have

$$|M_{n\wedge T}| \leq \max(a, -b) + T.$$

Using that T is in L^1 (as observed in part (1)) we get that $(M_{n\wedge T})$ is uniformly integrable. So again optional stopping applies and we get

$$\mathrm{E}(S_T) - \mathrm{E}(T) = 0,$$

which implies, using part (2), that

$$\mathrm{E}(T) = \mathrm{P}(T_a < T_b)a + \mathrm{P}(T_b < T_a)b = -ab = a|b|.$$

Exercise 3. [R] Let $(M_n)_{n\geq 0}$ be a $(\mathcal{F}_n)_{n\geq 0}$ martingale and let T be a $(\mathcal{F}_n)_{n\geq 0}$ stopping time.

(1) Assume that $\mathrm{E}(T) < \infty$ and there there exists $K > 0$ such that a.s. we have

$$\mathrm{E}(|M_{n+1} - M_n|) \mid \mathcal{F}_n \leq K$$

for every $n \geq 0$. Show that $\mathrm{E}(M_T) = \mathrm{E}(M_0)$.
Hint. Justify that $|M_{T\wedge n}| \leq |M_0| + \sum_{i=0}^{\infty} |M_{i+1} - M_i| 1_{T>i}$ and use dominated convergence.

(2) Let $(X_n)_{n\geq 1}$ be iid L^1 real-valued random variables. Set $S_0 = 0$, $S_n = X_1 + \cdots + X_n$ for $n \geq 1$ and $\mathcal{F}_n = \sigma(S_i : 0 \leq i \leq n)$ for $n \geq 0$. Finally, let T be a $(\mathcal{F}_n)$-stopping time with $\mathrm{E}(T) < \infty$. Show that

$$\mathrm{E}(S_T) = \mathrm{E}(X_1)\mathrm{E}(T).$$

Solution.

(1) Let us prove

$$|M_{T\wedge n}| \leq |M_0| + \sum_{i=0}^{\infty} |M_{i+1} - M_i| 1_{T>i}. \tag{1}$$

Write

$$M_{T\wedge n} = M_0 + \sum_{i=0}^{n\wedge T-1} (M_{i+1} - M_i) \leq M_0 + \sum_{i=0}^{\infty} (M_{i+1} - M_i) 1_{T>i},$$

and we get (1) by triangular inequality.

Since $\mathrm{E}(T) < \infty$, it follows that $T < \infty$ a.s. As a consequence, $M_{T\wedge n}$ converges almost surely to M_T. In addition, by (1), we are in position to use dominated convergence since $|M_0| + \sum_{i=0}^{\infty} |M_{i+1} - M_i| 1_{T>i}$ is integrable. Indeed using the fact that $1_{T>i}$ is $\mathcal{F}_i$ measurable, write

$$\begin{aligned}
\mathrm{E}(|M_0|) + \sum_{i=0}^{\infty} \mathrm{E}(|M_{i+1} - M_i| 1_{T>i}) &= \mathrm{E}(|M_0|) + \sum_{i=0}^{\infty} \mathrm{E}(\mathrm{E}(|M_{i+1} - M_i)| \mid \mathcal{F}_i) 1_{T>i} \\
&\leq \mathrm{E}(|M_0|) + \sum_{i=0}^{\infty} K \mathrm{E}(1_{T>i}) \\
&= \mathrm{E}(|M_0|) + K\mathrm{E}(T) < \infty,
\end{aligned}$$

where we have used the fact that $\mathrm{E}(Z) = \sum_{i=1}^{\infty} \mathrm{P}(Z \geq i)$ for every non-negative integer valued random variable Z. We thus get $\mathrm{E}(M_{T\wedge n}) \to \mathrm{E}(M_T)$ as $n \to \infty$. Since $\mathrm{E}(M_{T\wedge n}) = \mathrm{E}(M_0)$ for every $n \geq 0$, we get the desired result.

(2) We use (1) with the martingale $M_n = S_n - \mathrm{E}(X_1)n$. We just have to check that there exists $K > 0$ such that a.s. we have $\mathrm{E}(|M_{n+1} - M_n||\mathcal{F}_n) \leq K$ for every $n \geq 0$. To this end write

$$\mathrm{E}(|M_{n+1} - M_n||\mathcal{F}_n) = \mathrm{E}(|X_{n+1} - \mathrm{E}(X_1)||\mathcal{F}_n) \leq 2\mathrm{E}(|X_1|).$$

Exercise 4. Let $(M_n)_{n\geq 0}$ be a uniformly integrable martingale with respect to a filtration $(\mathcal{F}_n)_{n\geq 0}$.

(1) Is it true that the collection $\{M_T : T \text{ stopping time with respect to } (\mathcal{F}_n)_{n\geq 0}\}$ is uniformly integrable?

(2) Let T be a stopping time. Is it true that $(M_{n\wedge T})_{n\geq 0}$ is a uniformly integrable martingale? Justify your answer.

Solution.

(1) Yes it is true. It follows from the following two facts seen in the lecture : $M_T = \mathrm{E}(Z|\mathcal{F}_T)$ and for any collection of σ-fields $(\mathcal{A}_i)_{i\in I}$ the collection $(\mathrm{E}(Z|\mathcal{A}_i)_{i\in I}$ is uniformly integrable.

(2) Yes it is true, as a consequence of (1) since $n \wedge T$ is a stopping time for every $n \geq 0$.