PROBABILITY THEORY (D-MATH)
EXERCISE SHEET 2

Exercise 1. Give an example of a subsequence $(n(k))_{k \geq 1}$ such that
$$X_{n(k)} \xrightarrow{a.s.} 0,$$
where

(i) $(X_n)_{n \geq 1}$ is iid with $X_1 \sim \text{Ber}(1/n)$.

(ii) $(X_n)_{n \geq 1}$ is the typesetter sequence.

Exercise 2 [R]. Let (E, d) and (E', d') be metric spaces. Let $(X_n)_{n \geq 1}$ and X be random variables taking values in E.

(i) (Subsubsequence lemma) Show that X_n converges to X in probability if and only if for every subsequence $(n(k))_{k \geq 1}$ there exists a subsubsequence $(n(k(l)))_{l \geq 1}$ such that $X_{n(k(l))}$ converges to X almost surely as $l \to \infty$.

(ii) (Continuous mapping) Let $f : E \to E'$ be a continuous function. First, suppose $X_n \to X$ a.s. and show that $f(X_n) \to f(X)$ a.s. Next, suppose $X_n \to X$ in probability and show that $f(X_n) \to f(X)$ in probability.

Exercise 3 [R]. Let $(Y_n)_{n \geq 1}$ be a sequence of independent random variables such that $Y_n \sim \text{Exp}(\lambda_n)$, where $(\lambda_n)_{n \geq 1}$ is a sequence of positive real numbers such that $\lambda_n \to \infty$ as $n \to \infty$.

(i) Show that $Y_n \to 0$ in probability.

(ii) Let $\lambda_n = 10 \log n$. Does Y_n converge to 0 almost surely?

(iii) Let $\lambda_n = (\log n)^2$. Does Y_n converge to 0 almost surely?

Exercise 4. Define the space of functions
$$L^0 = \{X : \Omega \to E \text{ measurable}\}/ \sim,$$
where the equivalence relation $\sim$ is defined by
$$X \sim Y \iff X = Y \ a.s.$$

(i) Show that $D(X, Y) = \text{E}\big(1 \wedge d(X, Y)\big)$ defines a metric on L^0.

(ii) Assume E is complete. Show that (L^0, D) is complete.

Exercise 5. Let $(X_n)_{n \geq 1}$ be an iid sequence of random variables with $\text{E}(|X_1|) < \infty$. Define
$$S_n = \sum_{i=1}^{n} X_i X_{i+1}.$$
Show that S_n/n converges almost surely.

Submission of solutions. Hand in your solutions by 18:00, 04/10/2024 following the instructions on the course website
$$\texttt{https://metaphor.ethz.ch/x/2024/hs/401-3601-00L/}$$
Note that only the exercises marked with [R] will be corrected.