PROBABILITY THEORY (D-MATH)
EXERCISE SHEET 2 – SOLUTION

Exercise 1. Give an example of a subsequence $(n(k))_{k\geq 1}$ such that

$$X_{n(k)} \xrightarrow{a.s.} 0,$$

where

(i) $(X_n)_{n\geq 1}$ is iid with $X_1 \sim \text{Ber}(1/n)$.
(ii) $(X_n)_{n\geq 1}$ is the typesetter sequence.

Solution.

We use the criterion for almost sure convergence in section 5 of chapter 2 of the lecture notes.

(i) Consider the subsequence $n(k) = 2^k$. Let $\epsilon > 0$. Then

$$\sum_{k\geq 1} \mathrm{P}(X_{n(k)} \geq \epsilon) = \sum_{k\geq 1} \mathrm{P}(X_{n(k)} = 1) = \sum_{k\geq 1} 2^{-k} < \infty.$$

So by the criterion $X_{n(k)} \xrightarrow{a.s.} 0$.

(ii) Consider the subsequence $n(k) = 2^k$. Let $\epsilon > 0$. Then

$$\sum_{k\geq 1} \mathrm{P}(X_{n(k)} \geq \epsilon) = \sum_{k\geq 1} \mathrm{P}(\omega \in [0, 2^{-k}]) = \sum_{k\geq 1} 2^{-k} < \infty.$$

So by the criterion $X_{n(k)} \xrightarrow{a.s.} 0$.

Exercise 2 [R]. Let (E,d) and (E',d') be metric spaces. Let $(X_n)_{n\geq 1}$ and X be random variables taking values in E.

(i) (Subsubsequence lemma) Show that X_n converges to X in probability if and only if for every subsequence $(n(k))_{k\geq 1}$ there exists a subsubsequence $(n(k(l))_{l\geq 1}$ such that $X_{n(k(l))}$ converges to X almost surely as $l\to\infty$.

(ii) (Continuous mapping) Let $f : E \to E'$ be a continuous function. First, suppose $X_n \to X$ a.s. and show that $f(X_n)\to f(X)$ a.s. Next, suppose $X_n \to X$ in probability and show that $f(X_n)\to f(X)$ in probability.

Solution.

(i) ($\Rightarrow$) Suppose X_n converges to X in probability and let $(n(k))_{k\geq 1}$ be a subsequence. We also have that $X_{n(k)}$ converges to X in probability so the Proposition in section 7 of chapter 2 in the notes implies that we can extract a further subsequence $n(k(l))$ such that $X_{n(k(l))} \xrightarrow{a.s.} X$.

($\Leftarrow$) For a contradiction, suppose X_n does not converge to X in probability. Then there exist $\epsilon, \delta > 0$ and a subsequence $(n(k))_{k\geq 1}$ such that for all $k\geq 1$,

$$\mathrm{P}(|X_{n(k)} - X_n| \geq \epsilon) > \delta.$$

But then X_n cannot converge to X almost surely along any subsequence of $n(k)$, which contradicts the assumption.

(ii) First, suppose $X_n \to X$ a.s. Let $\omega \in \Omega$ be such that $X_n(\omega)\to X(\omega)$. Since f is continuous we have that $f(X_n(\omega)) \to f(X(\omega))$. So

$$\{\omega : X_n(\omega) \to X(\omega)\} \subset \{\omega : f(X_n(\omega)) \to f(X(\omega))\}.$$

So

$$\mathrm{P}(\{\omega : f(X_n(\omega)) \to f(X(\omega))\}) = 1.$$

So $f(X_n)\to f(X)$ a.s.

Second, suppose $X_n \to X$ in probability. We use the characterisation in exercise $2(i)$ to show that $f(X_n)$ converges to $f(X)$ in probability. Let $n(k)$ be a subsequence. By the ($\Rightarrow$) of the characterisation, we find a subsubsequence $n(k(l))$ such that $X_{n(k(l))} \to X$ almost surely as $l\to\infty$. By the previous part of this exercise, we get that $f(X_{n(k(l))}) \to f(X)$ almost surely as $l \to \infty$. So by ($\Leftarrow$) of the characterisation, $f(X_n)\to f(X)$ in probability.

Exercise 3 [R]. Let $(Y_n)_{n\geq 1}$ be a sequence of independent random variables such that $Y_n \sim \text{Exp}(\lambda_n)$, where $(\lambda_n)_{n\geq 1}$ is a sequence of positive real numbers such that $\lambda_n \to \infty$ as $n \to \infty$.

(i) Show that $Y_n \to 0$ in probability.
(ii) Let $\lambda_n = 10 \log n$. Does Y_n converge to 0 almost surely?
(iii) Let $\lambda_n = (\log n)^2$. Does Y_n converge to 0 almost surely?

Solution.

(i) Let $\epsilon > 0$. Then $\mathrm{P}(|Y_n| \geq \epsilon) = \exp(-\lambda_n)$, which converges to 0.
(ii) We show that Y_n does not converge almost surely to 0 in this case. Observe that

$$\sum_{n\geq 1} \mathrm{P}(Y_n > 1/20) = \sum_{n\geq 1} \exp\left(-\frac{\log n}{2}\right) = \sum_{n\geq 1} 1/\sqrt{n} = \infty.$$

Since the Y_n's are independent, the second Borel-Cantelli lemma implies that $\{Y_n > 1/20\}$ occurs infinitely often almost surely, which shows that Y_n does not converge to 0 almost surely.
(iii) Let $\lambda_n = (\log n)^2$. We show that Y_n converges to 0 almost surely by checking the criterion for almost sure convergence section 5 of chapter 2 of the lecture notes. Fix $\epsilon > 0$. Then

$$\sum_{n\geq 1} \mathrm{P}(Y_n > \epsilon) = \sum_{n\geq 1} \exp\left(-\epsilon \log^2 n\right) < \infty.$$

Exercise 4. Define the space of functions

$$L^0 = \{X : \Omega \to E \text{ measurable}\}/\sim,$$

where the equivalence relation $\sim$ is defined by

$$X \sim Y \iff X = Y \; a.s.$$

(i) Show that $D(X,Y) = \mathrm{E}\big(1 \wedge d(X,Y)\big)$ defines a metric on L^0.

(ii) Assume E is complete. Show that (L^0, D) is complete.

Solution.

(i) First, since $1 \wedge d(x,y) \geq 0$ for all $x, y \in E$, $D(X,Y) \geq 0$ for all $X, Y \in L^0$. Next, we show

$$D(X,Y) = 0 \iff X = Y \; a.s.$$

If $X = Y a.s.$ then $d(X,Y) a.s.$, so $D(X,Y) = 0$. For the other direction, suppose $X \neq Y$. By defininition, $\mathrm{P}(X \neq Y) > 0$. Writing

$$\mathrm{P}(X \neq Y) > 0 = \bigcap_{n \geq 1} \{\mathrm{P}(d(X,Y) > 1/n)\},$$

we see that there exists $n > 0$ such that $\mathrm{P}(d(X,Y) > 1/n)$. So $D(X,Y) = \mathrm{E}[1 \wedge D(X,Y)] \geq \mathrm{P}(d(X,Y) > 1/n)/n > 0$. Second, since d is symmetric, so is D. Third, let $X, Y, Z \in L^0$. By the triangle inequality for d we have for all ω

$$\begin{aligned} 1 \wedge d(X(\omega), Z(\omega)) &\leq 1 \wedge \big(d(X(\omega), Y(\omega)) + d(Y(\omega), Z(\omega))\big) \\ &\leq 1 \wedge d(X(\omega), Y(\omega)) + 1 \wedge d(X(\omega), Z(\omega)). \end{aligned}$$

Taking expectations proves the triangle inequality for D.

(ii) Assume E is complete. Let $(X_n)_{n \geq 1}$ be a Cauchy sequence in (L^0, D). We can find a subsequence of random variables $(Y_k = X_{n(k)})_{k \geq 1}$ such that that for all $k \geq 1$,

$$D(Y_k, Y_{k+1}) < 2^{-k}.$$

Then

$$\mathrm{E}\left(\sum_{k \geq 1} d(Y_k, Y_{k+1}) \wedge 1\right) < \infty,$$

which implies $\sum_{k \geq 1} d(Y_k, Y_{k+1}) \wedge 1 < \infty$ a.s. and therefore also $\sum_{k \geq 1} d(Y_k, Y_{k+1}) < \infty$ a.s. (because by Borel-Cantelli I, a.s. there are only finitely many values of k with $D(Y_k, Y_{k+1}) > 1$). So a.s. $(Y_k)_{k \geq 1}$ is a Cauchy sequence in E; using completemeness of E define Y to be the limit of the sequence under this event and 0 on its complement. By construction, we have

$$d(Y_n, Y) \wedge 1 \to 0 \; a.s.$$

So by dominated convergence $D(Y_n, Y) \to 0$. Since (X_n) is Cauchy and a subsequence $X_{n(k)} \to Y$, $X_n \to Y$ as well.

Exercise 5. Let $(X_n)_{n\geq 1}$ be an iid sequence of random variables with $\mathrm{E}(|X_1|) < \infty$. Define

$$S_n = \sum_{i=1}^{n} X_i X_{i+1}.$$

Show that S_n/n converges almost surely.

Solution.

We let

$$M_n^+ = \frac{1}{n}\sum_{i=1}^{n} X_{2i}X_{2i+1} \quad \text{and} \quad M_n^- = \frac{1}{n}\sum_{i=1}^{n} X_{2i-1}X_{2i} \ .$$

Since the sequences $(X_{2i}X_{2i+1})_{i\geq 1}$ and $(X_{2i-1}X_{2i})_{i\geq 1}$ are i.i.d. and $\mathrm{E}(X_{2i}X_{2i+1}) = \mathrm{E}(X_{2i-1}X_{2i}) = \mathrm{E}(X)^2$, the strong law of large numbers implies that

$$M_n^+ \to \mathrm{E}(X)^2 \quad \text{and} \quad M_n^- \to \mathrm{E}(X)^2 \quad \text{as } n \to \infty \text{ a.s.}$$

Note that $M_{2n} = M_{n-1}^+ \cdot (1/2 - 1/(2n)) + M_n^-/2$ and $M_{2n+1} = (M_n^+ + M_n^-) \cdot n/(2n+1)$, so we deduce that $M_n \to \mathrm{E}(X)^2$ as $n \to \infty$ almost surely.