PROBABILITY THEORY (D-MATH)
EXERCISE SHEET 3

Exercise 1. Let $(u_n)_{n\geq 1}$ and c be real numbers. Suppose $\lim_{n\to\infty} u_n = c$. Show that

$$\lim_{n\to\infty} \frac{u_1 + \cdots + u_n}{n} = c.$$

Exercise 2. [R] Let $(X_n)_{n\geq 1}$ be pairwise independent, positive, identically distributed random variables with $E(X_1) = \infty$. Show that

$$\frac{X_1 + \cdots + X_n}{n} \xrightarrow[n\to\infty]{a.s.} \infty.$$

Hint: for $a > 0$ consider the random variables $\min(X_n, a)$.

Exercise 3. [Hard] Give an example of an iid sequence $(X_n)_{n\geq 1}$ such that almost surely

$$\limsup_{n\to\infty} \frac{X_1 + \cdots + X_n}{n} = \infty \quad \text{and} \quad \liminf_{n\to\infty} \frac{X_1 + \cdots + X_n}{n} = -\infty.$$

Exercise 4. Let $(X_n)_{n\geq 1}$ be an iid sequence of random variables that are uniformly distributed in unit ball $\{x \in \mathbb{R}^2 : \|x\|_2 \leq 1\}$. Define $(Z_n)_{n\geq 1}$ inductively by $Z_0 = (1,0)$ and $Z_{n+1} = \|X_{n+1}\|_2 \cdot Z_n$.

(i) Show that there exists $c \in \mathbb{R}$ such that

$$\frac{\log \|Z_n\|_2}{n} \xrightarrow[n\to\infty]{a.s.} c.$$

(ii) Compute the value of c.

(iii) What is the limit when $Z_0 = (2, 2)$?

Exercise 5. [R]

(i) Show that a family of random variables $(X_i)_{i\in I}$ defined on a probability space $(\Omega, \mathcal{F}, P)$. Show this family is uniformly integrable if and only if it is bounded in L^1 (that is, there exists $M \in \mathbb{R}$ such that for all $i \in I$, $E(|X_i|) \leq M$) and for all $\epsilon > 0$ there exists $\delta > 0$ such that for all $A \in \mathcal{F}$ with $P(A) \leq \delta$ and all $i \in I$ we have $E(|X_i| 1_A) \leq \epsilon$.

(ii) Let $(X_i)_{i\in I}$ and $(Y_j)_{j\in J}$ be two uniformly integrable families of random variables. Show that $(X_i + Y_j)_{(i,j)\in I\times J}$ is uniformly integrable.

Submission of solutions. Hand in your solutions by 18:00, 11/10/2024 following the instructions on the course website

Note that only the exercises marked with [R] will be corrected.