PROBABILITY THEORY (D-MATH)
EXERCISE SHEET 3 – SOLUTION

Exercise 1. Let $(u_n)_{n\geq 1}$ and c be real numbers. Suppose $\lim_{n\to\infty} u_n = c$. Show that

$$\lim_{n\to\infty} \frac{u_1 + \cdots + u_n}{n} = c.$$

Solution. Let $\epsilon > 0$ and let m be such that for all $n \geq m$ we have $|u_n - c| < \epsilon$. Then we have

$$\begin{aligned}
\limsup_{n\to\infty} \left| \frac{u_1 + \cdots + u_n}{n} - c \right| &= \limsup_{n\to\infty} \frac{1}{n} \left| u_1 + \cdots + u_n - nc \right| \\
\text{(triangle inequality)} \qquad &\leq \limsup_{n\to\infty} \left| \frac{u_1 + \cdots + u_m}{n} \right| + \limsup_{n\to\infty} \frac{(n-m)\epsilon}{n} \\
&\leq \epsilon.
\end{aligned}$$

Since ϵ was arbitrary, this completes the proof.

Exercise 2 [R]. Let $(X_n)_{n\geq 1}$ be pairwise independent, positive, identically distributed random variables with $\mathrm{E}(X_1) = \infty$. Show that

$$\frac{X_1 + \cdots + X_n}{n} \xrightarrow[n\to\infty]{a.s.} \infty.$$

Hint: for $a > 0$ consider the random variables $\min(X_n, a)$.

Solution.

Let $a > 0$ and observe that $(X_n \wedge a)_{n\geq 1}$ are pairwise independent, positive, identical random variables all bounded by a. By the strong law of large numbers we have almost surely that

$$\mathrm{E}(X_1 \wedge a) = \liminf_{n\to\infty} \frac{X_1 \wedge a + \cdots + X_n \wedge a}{n} \leq \liminf_{n} \frac{X_1 + \cdots + X_n}{n}.$$

Now, by monotone convergence $\mathrm{E}(X_1 \wedge a) \to \infty$ as $a \to \infty$. Taking a countable sequence $a \to \infty$ shows that almost surely

$$\liminf_{n\to\infty} \frac{X_1 + \cdots + X_n}{n} = \infty,$$

as desired.

Remark. The hypothesis that the X_n's are positive is included because otherwise $\mathrm{E}(X_n)$ is not well-defined in general.

Exercise 3. [Hard] Give an example of an iid sequence $(X_n)_{n\geq 1}$ such that almost surely

$$\limsup_{n\to\infty} \frac{X_1 + \cdots + X_n}{n} = \infty \quad \text{and} \quad \liminf_{n\to\infty} \frac{X_1 + \cdots + X_n}{n} = -\infty.$$

Solution. Let $(Y_n)_{n\geq 1}$ be a sequence of iid positive random variables with $\mathrm{E}(Y_1) = \infty$. Let $(Z_n)_{n\geq 1}$ be an iid sequence, independent of $(Z_n)_{n\geq 1}$, with $\mathrm{P}(Z_1 = 1) = \mathrm{P}(Z_1 = -1) = 1/2$. For $n \geq 1$ define $X_n = Y_n \cdot Z_n$. For $n \geq 1$, $S_n = X_1 + \cdots + X_n$. We show that almost surely

$$\limsup_{n\to\infty} S_n/n = \infty \quad \text{and} \quad \liminf_{n\to\infty} S_n/n = -\infty.$$

We claim that

$$\forall a > 0 \quad \mathrm{P}(\forall n \geq 1 \quad X_n \in [-a, a]) = 0.$$

We complete the proof assuming this claim and then prove the claim. By the claim we have

$$\mathrm{P}\left(\bigcup_{a\in\mathbb{N}} \{\forall n \geq 1 \quad S_n/n \in [-a, a]\} \right) = 0,$$

or in other words

$$\mathrm{P}(|S_n/n| \text{ is bounded}) = 0.$$

This is equivalent to

$$\mathrm{P}(\{\limsup S_n/n = \infty\} \cup \{\liminf S_n/n = -\infty\}) = 1.$$

Therefore, in particular, one of the events in the union above has positive probability. By symmetry (more precisely, using the fact that (X_n) and $(-X_n)$ have the same distribution), we see that the events in the union have the same probability. So both events have positive probability. Finally, observe that both events are in the tail sigma algebra of $(X_n)_{n\geq 1}$, an independent sequence. By the Kolmogorov 0-1 law, we conclude that both events have probability 1, as required.

Now, we prove the claim. Fix $a > 0$. For $n \geq 1$ define the event $A_n = \{|X_n| \geq 2an\}$. Since $\mathrm{E}(|X_1| = \infty)$, we also have

$$\sum_{n \geq 1} \mathrm{P}(A_n) = \infty.$$

Since the A_n's are independent, the second Borel-Cantelli lemma implies, in particular, that one of the A_n's occur almost surely. Now, suppose $\omega \in A_n$. We show that $\omega \notin \{\forall n \geq 1 \quad S_n/n \in [-a, a]\}$. Indeed, if $|S_{n-1}(\omega)| > (n-1)a$, we are done and if $|S_{n-1}(\omega)| \leq (n-1)a$, then $|S_n(\omega)| > na$, so we are done anyway. This proves the claim and completes the proof.

Exercise 4. Let $(X_n)_{n\geq 1}$ be an iid sequence of random variables that are uniformly distributed in unit ball $\{x \in \mathbb{R}^2 : \|x\|_2 \leq 1\}$. Define $(Z_n)_{n\geq 1}$ inductively by $Z_0 = (1,0)$ and $Z_{n+1} = \|X_{n+1}\|_2 \cdot Z_n$.

(i) Show that there exists $c \in \mathbb{R}$ such that

$$\frac{\log \|Z_n\|_2}{n} \xrightarrow[n\to\infty]{a.s.} c.$$

(ii) Compute the value of c.

(iii) What is the limit when $Z_0 = (2,2)$?

Solution.

(i) For simplicity we drop the subscript in $\|\cdot\|_2$. Observe that for $n \geq 1$

$$\|Z_n\| = \|X_1\|\|X_2\|\cdots\|X_n\|.$$

So

$$\log \|Z_n\| = \log \|X_1\| + \cdots + \log \|X_n\|.$$

One can check that $\|X_1\|$ has density given by $2r1_{0\leq r\leq 1}$. Therefore, $\|X_i\| \in (0,1]$ a.s., so $\log \|X_i\|$ is a random variable taking values in $(-\infty, 0]$, with expectation given by

$$\mathrm{E}(\log \|X_1\|) = \int_0^1 2r \log r \, dr = -1/2.$$

The desired conclusion now follows from the strong law of large numbers.

(ii) The strong law of large numbers says that $c = \mathrm{E}(\|X_1\|) = -1/2$.

(iii) In this case, we see that for all $n \geq 1$,

$$\|Z_n\| = \sqrt{8}\|X_1\|\|X_2\|\cdots\|X_n\|,$$

so

$$\log \|Z_n\| = \log \sqrt{8} + \log \|X_1\| + \cdots + \log \|X_n\|.$$

So the limit is $-1/2$ in this case as well.

Remark: This exercise shows that for any starting point X_0,

$$\|Z_n\| = \exp(-n/2 + o(n)).$$

Exercise 5. [R]

(i) Show that a family of random variables $(X_i)_{i\in I}$ defined on a probability space $(\Omega, \mathcal{F}, \mathrm{P})$. Show this family is uniformly integrable if and only if it is bounded in L^1 (that is, there exists $M \in \mathbb{R}$ such that for all $i \in I$, $\mathrm{E}(|X_i|) \leq M$) and for all $\epsilon > 0$ there exists $\delta > 0$ such that for all $A \in \mathcal{F}$ with $\mathrm{P}(A) \leq \delta$ and all $i \in I$ we have $\mathrm{E}(|X_i|1_A) \leq \epsilon$.

(ii) Let $(X_i)_{i\in I}$ and $(Y_j)_{j\in J}$ be two uniformly integrable families of random variables. Show that $(X_i + Y_j)_{(i,j)\in I\times J}$ is uniformly integrable.

Solution.

(i) First, suppose $= (X_i)_{i\in I}$ is uniformly integrable. To show that (X_i) is bounded in L^1 choose $a > 0$ such that for all $i \in I$, $\mathrm{E}(X_i 1_{|X_i|\geq a}) < 1$ to get

$$\forall i \in I \quad \mathrm{E}(X_i) = \mathrm{E}(X_i 1_{|X_i|<a}) + \mathrm{E}(X_i 1_{|X_i|\geq a}) \leq a + 1.$$

Fix $\epsilon > 0$. Choose a such that for all $i \in I$, $\mathrm{E}(X_i 1_{|X_i|\geq a}) < \epsilon/2$. Let $\delta = \epsilon/(2a)$ and let $A \subset \mathcal{F}$ be such that $\mathrm{P}(A) \leq \delta$. Then for all $i \in I$ we have

$$\begin{aligned}
\mathrm{E}(X_i 1_A) &= \mathrm{E}(X_i 1_A 1_{|X_i|<a}) + \mathrm{E}(X_i 1_A 1_{|X_i|\geq a}) \\
&\leq a \cdot \mathrm{E}(1_A 1_{|X_i|<a}) + \mathrm{E}(X_i 1_{|X_i|\geq a}) \\
&\leq a \cdot \mathrm{P}(A) + \epsilon/2 \\
&\leq \epsilon/2 + \epsilon/2 = \epsilon.
\end{aligned}$$

Now, we prove the other direction. Let M be the uniform bound on the L^1 norms of X_i, that is for all $i \in I$ $\mathrm{E}(|X_i|) \leq M$. Fix $\epsilon > 0$. Choose $\delta > 0$ such that for all $A \in \mathcal{F}$ with $\mathrm{P}(A) < \delta$ we have $\mathrm{E}(|X_i|1_A) \leq \epsilon$ for all $i \in I$. Let $a = M/\delta$ and note that by Markov's inequality for all $i \in I$,

$$\mathrm{P}(|X_i| \geq a) \leq \mathrm{E}(|X_i|)/a \leq \delta.$$

So we get for all $i \in I$,

$$\mathrm{E}(|X_i| 1_{|X_i|\geq a}) \leq \epsilon,$$

which proves that $(X_i)_{i\in I}$ is uniformly integrable.

(ii) To show that $(X_i + Y_j)_{(i,j)\in I\times J}$ is uniformly integrable, we verify the criterion in the previous part. First, we check that $(X_i + Y_j)_{(i,j)\in I\times J}$ is bounded in L^1. Let M be such that for all $i \in I$, $\mathrm{E}(|X_i|) < M$ and let N be such that for all $j \in J$, $\mathrm{E}(|Y_j|) < N$. Then for all $i \in I$ and $j \in J$ we have $\mathrm{E}(|X_i + Y_j|) \leq \mathrm{E}(|X_i|) + \mathrm{E}(|Y_j|) < M + N$.

Next, fix $\epsilon > 0$ using that (X_i) and (Y_i) are uniformly integrable, choose $\delta_1 > 0$ for all $A \in \mathcal{F}$ with $\mathrm{P}(A) \leq \delta_1$ and all $i \in I$ we have

$$\mathrm{E}(|X_i|1_A) \leq \epsilon/2$$

and choose δ_2 such that for all $A \in \mathcal{F}$ with $\mathrm{P}(A) \leq \delta$ and all $j \in J$ we have

$$\mathrm{E}(|Y_j|1_A) \leq \epsilon/2.$$

Let $\delta = \delta_1 \wedge \delta_2$ and let $A \in \mathcal{F}$ be such that $\mathrm{P}(A) \leq \delta$. Then we have

$$\forall (i,j) \in I \times J \quad \mathrm{E}(|X_i + Y_j|1_A) \leq \mathrm{E}(|X_i|1_A) + \mathrm{E}(|Y_j|1_A) \leq \epsilon/2 + \epsilon/2 = \epsilon,$$

which completes the proof.