PROBABILITY THEORY (D-MATH)
EXERCISE SHEET 4

Exercise 1. [R] Let $(X_n)_{n\geq 1}, X$ be random variables such that $X_n \xrightarrow{\mathrm{P}} X$ as $n \to \infty$. Show that the following are equivalent.

(1) $(X_n)_{n\geq 1}$ is uniformly integrable.
(2) $(X_n), X$ are all in L^1 and $\mathrm{E}[|X_n|] \to \mathrm{E}[|X|]$ as $n \to \infty$.

Exercise 2. [R] Give an example of a sequence of random variables $(X_n)_{n\geq 1}$ that is not uniformly integrable and a random variable X such that

$$X_n \xrightarrow{\mathrm{P}} X \quad \text{and} \quad \mathrm{E}(X_n) \to \mathrm{E}(X),$$

as $n \to \infty$.

Exercise 3. Let $(X_n)_{n\geq 1}$ be a sequence of iid real-valued random variables. Show that if $\mathrm{E}(|X1|) < \infty$, then the sequence $(\max(X_1, \dots, X_n)/n)_{n\geq 2}$ is uniformly integrable. Is the converse true?

Exercise 4. Consider the probability space defined by $\Omega = \{1, 2, 3, 4, 5, 6\}$, $\mathcal{F} = 2^\Omega$, and

$$\forall A \in \mathcal{F} \quad \mathrm{P}(A) = |A|/6.$$

Define random variables X and Y by

$$X(\omega) = \omega \pmod 2 \quad \text{and} \quad Y(\omega) = \omega \pmod 3.$$

Is $\sigma(X, Y) = \sigma(X) \cup \sigma(Y)$?

Exercise 5. [R] Let $p \in [0, 1]$. Let $(X_n)_{n\geq 1}, Y$ be independent random variables with distributions specified as follows: $(X_n)_{n\geq 1}$ is an iid sequence of random variables with $\mathrm{P}(X_1 = 1) = 1 - \mathrm{P}(X_1 = -1) = p$ and $\mathrm{P}(Y = 1) = \mathrm{P}(Y = -1) = 1/2$. For $n \geq 1$, define $Z_n = X_n \cdot Y$. For which values of $p \in [0, 1]$ is the tail sigma algebra of $(Z_n)_{n\geq 1}$ trivial?

Submission of solutions. Hand in your solutions by 18:00, 18/10/2024 following the instructions on the course website

`https://metaphor.ethz.ch/x/2024/hs/401-3601-00L/`

Note that only the exercises marked with [R] will be corrected.