PROBABILITY THEORY (D-MATH)
EXERCISE SHEET 4 – SOLUTION

Exercise 1. [R] Let $(X_n)_{n\geq 1}, X$ be random variables such that $X_n \xrightarrow{\mathrm{P}} X$ as $n \to \infty$. Show that the following are equivalent.

(1) $(X_n)_{n\geq 1}$ is uniformly integrable.
(2) $(X_n), X$ are all in L^1 and $\mathrm{E}[|X_n|] \to \mathrm{E}[|X|]$ as $n \to \infty$.

Solution. First, suppose that $(X_n)_{n\geq 1}$ is uniformly integrable. Then, by the main theorem in section 6 of chapter 3, we have that $X_n \xrightarrow{L^1} X$. So, by the triangle inequality, we have

$$|\mathrm{E}(|X| - |X_n|)| \leq \mathrm{E}(|X - X_n|) \to 0,$$

as required.

Second, suppose that $\mathrm{E}(|X_n|) \to \mathrm{E}(X)$. For $M \geq 1$, define the function $\psi_M : \mathbb{R}_{\geq 0} \to \mathbb{R}$ by

$$\phi_M(x) = \begin{cases} x & \text{if } x \in [0, M-1], \\ 0 & \text{if } x \in [M, \infty), \\ \text{linear} & \text{if } x \in [M-1, M]. \end{cases} \tag{1}$$

Fix $\epsilon > 0$. By dominated convergence $\mathrm{E}\big(\psi_M(|X|)\big) \to \mathrm{E}\big(|X|\big)$ as $m \to \infty$. So we can choose M so that

$$\mathrm{E}\big(|X|\big) - \mathrm{E}\big(\phi_M(|X|)\big) \leq \epsilon/2.$$

Next, for all $M \geq 1$, we have that $\psi_M(|X_n|) \xrightarrow{\mathrm{P}} \psi_M(|X|)$ as $n \to \infty$, since ψ_M is continuous. Since ψ_M is bounded, the implication we proved in the first part of this exercise gives that

$$\mathrm{E}\big(\psi_M(|X_n|\big) \to \mathrm{E}\big(\psi_M(|X|)\big) \text{ as } n \to \infty.$$

Using this and the assumption that $\mathrm{E}(|X_n|) \to \mathrm{E}(X)$, for n large enough we have

$$\begin{aligned} \mathrm{E}\big(|X_n| 1_{|X_n| \geq M}\big) &\leq \mathrm{E}\big(|X_n|\big) - \mathrm{E}\big(\psi_M(X_n)\big) \\ &\leq \mathrm{E}\big(|X|\big) - \mathrm{E}\big(\psi_M(|X|)\big) + \epsilon/2 \leq \epsilon. \end{aligned}$$

Choosing M larger if necessary we can ensure that $\mathrm{E}\big(|X_n| 1_{|X_n| \geq M}\big) \leq \epsilon$ for all $n \geq 1$, completing the proof.

Exercise 2. [R] Give an example of a sequence of random variables $(X_n)_{n\geq 1}$ that is not uniformly integrable and a random variable X such that

$$X_n \xrightarrow{\mathrm{P}} X \quad \text{and} \quad \mathrm{E}(X_n) \to \mathrm{E}(X),$$

as $n \to \infty$.

Solution. For $n \geq 2$ define

$$X_n = \begin{cases} n & \text{w.p. } 1/n, \\ -n & \text{w.p. } 1/n, \\ 0 & \text{w.p. } 1 - 2/n. \end{cases}$$

Then $X_n \xrightarrow{P} 0$ and $\mathrm{E}(X_n) = 0$ for all n, but (X_n) is not uniformly integrable.

Exercise 3. Let $(X_n)_{n\geq 1}$ be a sequence of iid real-valued random variables. Show that if $\mathrm{E}(|X1|) < \infty$, then the sequence $(\max(X_1, \ldots, X_n)/n)_{n\geq 2}$ is uniformly integrable. Is the converse true?

Solution. To show that the sequence $(\max(X_1, \ldots, X_n)/n)_{n\geq 2}$ is uniformly integrable note $n \geq 1$,

$$\max(X_1, \ldots, X_n) \leq \frac{|X_1| + \cdots + |X_n|}{n}.$$

We showed that the sequence is

$$\left(\frac{|X_1| + \cdots + |X_n|}{n}\right)_{n\geq 1}$$

is uniformly integrable in section 3 of chapter 6 when we proved the L^1 version of the law of large numbers, which completes the proof.

The converse is true if the random variables are non-negative. In such cases, $0 \leq X_1/2 \leq \max(X_1, X_2)/2$, and as a result, $\mathrm{E}(|X_1|) < \infty$.

In general, no! For example, if we take X_1 to be a random variable with density given by

$$\frac{1}{|x|^2} 1_{x\leq -1},$$

we can observe that $\max(X_1, \ldots, X_n)$ has density

$$\frac{n}{|x|^{n+1}} 1_{x\leq -1},$$

and as a result,

$$\mathrm{E}\left(\left|\frac{\max(X_1, \ldots, X_n)}{n}\right|^{3/2}\right) = \frac{1}{n^{3/2}} \int_1^{\infty} \frac{n x^{3/2}}{x^{n+1}} \mathrm{d}x = \frac{1}{n^{3/2}} \frac{2n}{2n-3}.$$

Hence, the sequence $\left(\frac{\max(X_1,\ldots,X_n)}{n}\right)_{n\geq 2}$ is bounded in $L^{3/2}$, so it is uniformly integrable, even though X_1 is not integrable.

Exercise 4. Consider the probability space defined by $\Omega = \{1, 2, 3, 4, 5, 6\}$, $\mathcal{F} = 2^\Omega$, and

$$\forall A \in \mathcal{F} \quad \mathrm{P}(A) = |A|/6.$$

Define random variables X and Y by

$$X(\omega) = \omega \pmod 2 \quad \text{and} \quad Y(\omega) = \omega \pmod 3.$$

Is $\sigma(X, Y) = \sigma(X) \cup \sigma(Y)$?

Solution. It is easy to check that

$$\sigma(X) = \big\{\emptyset, \Omega, \{2, 4, 6\}, \{1, 3, 5\}\big\}$$

and

$$\sigma(Y) = \big\{\emptyset, \Omega, \{1, 4\}, \{2, 5\}, \{3, 6\}, \{1, 2, 4, 5\}, \{1, 3, 4, 6\}, \{2, 3, 5, 6\}\big\}.$$

So $\sigma(X) \cup \sigma(Y)$ is not a sigma algebra, so it cannot be equal to $\sigma(X, Y)$. In fact, it is easy to check that $\sigma(X, Y) = 2^\Omega$.

Exercise 5. [R] Let $p \in [0,1]$. Let $(X_n)_{n\geq 1}, Y$ be independent random variables with distributions specified as follows: $(X_n)_{n\geq 1}$ is an iid sequence of random variables with $\mathrm{P}(X_1 = 1) = 1 - \mathrm{P}(X_1 = -1) = p$ and $\mathrm{P}(Y = 1) = \mathrm{P}(Y = -1) = 1/2$. For $n \geq 1$, define $Z_n = X_n \cdot Y$. For which values of $p \in [0,1]$ is the tail sigma algebra of $(Z_n)_{n\geq 1}$ trivial?

Solution. We claim that the tail of (Z_n) is trivial if and only if $p = 1/2$. First, if $p = 1/2$ then (Z_n) is an iid sequence, so by Kolmogorov's $0-1$ law, the tail is trivial. Suppose $p > 1/2$. By the strong law of large numbers,

$$\frac{Z_1 + \cdots + Z_n}{n} = Y \cdot \left(\frac{X_1 + \cdots + X_n}{n}\right) \xrightarrow{a.s.} Y(2p-1) \quad \text{as } n \to \infty.$$

Therefore, the tail event $\{\lim(Z_1 + \cdots + Z_n)/n > 0\}$ has probability $1/2$, showing that the tail sigma algebra of (Z_n) is non-trivial. An analogous argument in the case $p < 1/2$ completes the proof.

Remark. We say a sigma algebra $\mathcal{A}$ is trivial if for all $A \in \mathcal{A}$, $\mathrm{P}(A) = 0$ or $\mathrm{P}(A) = 1$.