PROBABILITY THEORY (D-MATH)
EXERCISE SHEET 5

Exercise 1. Let $\alpha, \beta > 0$ be real numbers. Let $X \sim \text{Poi}(\alpha)$ and $Y \sim \text{Poi}(\beta)$ be independent random variables. Show that $X + Y \sim \text{Poi}(\alpha + \beta)$.

Exericse 2. [R] This exercise shows that the tail of a random variable is determined by the behaviour of its characteristic function around zero. Let X be a real-valued random variable and let ϕ be its characterisitic function. Show that

$$\mathrm{P}(|X| > 2/u) \leq \frac{1}{u} \int_{-u}^{u} \big(1 - \phi(t)\big)\, dt.$$

Exercise 3. [R] Let X be a real-valued random variable such that its characteristic function $\phi_X \in L^1(\mathbb{R})$.

(i) Show that for all

$$\forall \psi \in \mathcal{C}_c^\infty \quad \mathrm{E}\big(\psi(X)\big) = \int_{\mathbb{R}} \psi(x) \int_{\mathbb{R}} \phi(t) e^{-itx}\, dt\, dx.$$

(ii) Deduce that X has a density.

Exercise 4. Let $X_0, X_1, \ldots$ be iid random variables with

$$\mathrm{P}(X_0 = 1) = \mathrm{P}(X_0 = -1) = 1/2.$$

For $n \geq 1$ define

$$Y_n = X_0 \cdots X_n.$$

Let

$$\mathcal{X} = \sigma(X_1, X_2, \ldots) \quad \text{and} \quad \mathcal{Y}_n = \sigma(Y_n, Y_{n+1}, \ldots).$$

The aim of this exercise is to show that

$$\bigcap_{n\geq 1} \sigma(\mathcal{X}, Y_n) \quad \text{and} \quad \sigma\Big(\mathcal{X}, \bigcap_{n\geq 1} \mathcal{Y}_n\Big)$$

are not equal.

(i) Show that $\sigma(X_0) \subset \sigma(\mathcal{X}, Y_n)$ for each $n \geq 1$.
(ii) Show that $\bigcap_{n\geq 1} \mathcal{Y}_n$ is trivial.
(iii) Show that $\sigma(X_0)$ is independent of $\sigma\big(\mathcal{X}, \bigcap_{n\geq 1} \mathcal{Y}_n\big)$. (Hint: check independence on a suitable π-system.)
(iv) Conclude.

Submission of solutions. Hand in your solutions by 18:00, 25/10/2024 following the instructions on the course website

`https://metaphor.ethz.ch/x/2024/hs/401-3601-00L/`

Note that only the exercises marked with [R] will be corrected.