PROBABILITY THEORY (D-MATH)
EXERCISE SHEET 5 – SOLUTION

Exercise 1. Let $\alpha, \beta > 0$ be real numbers. Let $X \sim \mathrm{Poi}(\alpha)$ and $Y \sim \mathrm{Poi}(\beta)$ be independent random variables. Show that $X + Y \sim \mathrm{Poi}(\alpha + \beta)$.

Solution. We compute the characteristic function of a $\mathrm{Poi}(\alpha)$ random variable as follows.

$$\begin{aligned}\phi_X(t) = \mathrm{E}(e^{itX}) &= \sum_{k\geq 0} \frac{e^{itk} e^{-\alpha} \alpha^k}{k!} \\ &= e^{-\alpha} \sum_{k\geq 0} \frac{\left(e^{it}\alpha\right)^k}{k!} \\ &= e^{-\alpha} e^{\alpha e^{it}} = e^{\alpha(e^{it}-1)}.\end{aligned}$$

Since X and Y are independent we get

$$\phi_{X+Y}(t) = \phi_X(t)\phi_Y(t) = e^{(\alpha+\beta)(e^{it}-1)},$$

which is the characteristic function of a $\mathrm{Poi}(\alpha+\beta)$ random variable. Since the characterstic function of a random variable characterises its law, the result follows.

Exericse 2. [R] This exercise shows that the tail of a random variable is determined by the behaviour of its characteristic function around zero. Let X be a real-valued random variable and let ϕ be its characterisitic function. Show that

$$\mathrm{P}(|X| > 2/u) \leq \frac{1}{u} \int_{-u}^{u} \big(1 - \phi(t)\big)\, dt.$$

Solution.

$$\begin{aligned}
\frac{1}{u} \int_{-u}^{u} \big(1 - \phi(t)\big)\, dt &= \frac{1}{u} \int_{-u}^{u} \int_{\mathbb{R}} (1 - e^{itx})\, d\mu_X(x)\, dt \\
\text{(Fubini for integrable functions)} \qquad &= \frac{1}{u} \int_{\mathbb{R}} \int_{-u}^{u} (1 - e^{itx})\, dt\, d\mu_X(x) \\
&= \int_{\mathbb{R}} 2 - \frac{e^{iux} - e^{-iux}}{iux}\, d\mu_X(x) \\
&= 2 \int_{\mathbb{R}} 1 - \frac{\sin ux}{ux}\, d\mu_X(x) \\
(\forall x \;\; \sin ux \leq ux) \qquad &\geq 2 \int_{|x|>2/u} 1 - \frac{\sin ux}{ux}\, d\mu_X(x) \\
(\forall x \; |\sin ux| \leq 1) \qquad &\geq 2 \int_{|x|>2/u} 1 - \frac{1}{|ux|}\, d\mu_X(x) \\
(|ux| > 2) \qquad &\geq \int_{|x|>2/u} d\mu_X(x) \\
&= \mathrm{P}(|X| > 2/u).
\end{aligned}$$

Exercise 3. [R] Let X be a real-valued random variable such that its characteristic function $\phi_X \in L^1(\mathbb{R})$.

(i) Show that for all

$$\forall \psi \in \mathcal{C}_c^\infty \quad \mathrm{E}\big(\psi(X)\big) = \frac{1}{2\pi} \int_{\mathbb{R}} \psi(x) \int_{\mathbb{R}} \phi(t) e^{-itx} \, dt \, dx.$$

(ii) Deduce that X has a density.

Solution.

(i) Let $\psi \in \mathcal{C}_c^\infty$. From section 3 of chapter 5 and using that $\phi_X \in L^1$ in the step with Fubini's theorem we get:

$$\begin{aligned} \mathrm{E}\big(\psi(X)\big) &= \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{\psi}(t) \overline{\phi_X(t)} \, dt \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} \overline{\phi_X(t)} \int_{\mathbb{R}} \psi(x) e^{ixt} \, dt \\ \text{(Fubini for integrable functions)} \quad &= \frac{1}{2\pi} \int_{\mathbb{R}} \psi(x) \int_{\mathbb{R}} \overline{\phi_X(t)} e^{ixt} \, dt \\ (t \mapsto -t) \quad &= \frac{1}{2\pi} \int_{\mathbb{R}} \psi(x) \int_{\mathbb{R}} \phi_X(t) e^{-ixt} \, dt. \end{aligned}$$

(ii) Setting

$$f(x) = \int_{\mathbb{R}} \phi_X(t) e^{-ixt} \, dt,$$

the formula we just proved shows that $f(x)$ is the density of X. (It is also true, moreover, that f is continuous and bounded.)

Exercise 4. Let $X_0, X_1, \ldots$ be iid random variables with

$$\mathrm{P}(X_0 = 1) = \mathrm{P}(X_0 = -1) = 1/2.$$

For $n \geq 1$ define

$$Y_n = X_0 \cdots X_n.$$

Let

$$\mathcal{X} = \sigma(X_1, X_2, \ldots) \quad \text{and} \quad \mathcal{Y}_n = \sigma(Y_n, Y_{n+1}, \ldots).$$

The aim of this exercise is to show that

$$\bigcap_{n\geq 1} \sigma(\mathcal{X}, Y_n) \quad \text{and} \quad \sigma\Big(\mathcal{X}, \bigcap_{n\geq 1} \mathcal{Y}_n\Big)$$

are not equal.

(i) Show that $\sigma(X_0) \subset \sigma(\mathcal{X}, Y_n)$ for each $n \geq 1$.
(ii) Show that $\bigcap_{n\geq 1} \mathcal{Y}_n$ is trivial.
(iii) Show that $\sigma(X_0)$ is independent of $\sigma\big(\mathcal{X}, \bigcap_{n\geq 1} \mathcal{Y}_n\big)$. (Hint: check independence on a suitable π-system.)
(iv) Conclude.

Solution.

(i) This follows from the formula $X_0 = Y_n X_1 X_2 \cdots X_n$.
(ii) Observe that $(Y_n)_{n\geq 1}$ is an independent sequence (actually iid) of random variables, so the Kolomogorov implies that its tail is trivial.
(iii) Consider the following family of subsets.

$$\mathcal{S} = \Big\{ A \cap B : A \in \mathcal{X}, B \in \bigcap_{n\geq 1} \mathcal{Y}_n \Big\}.$$

It is easy to see that this set is a $\pi-$system (that is, it is closed under finite intersections), and furthermore taking A or B to be Ω, we see that $\mathcal{S}$ contains both $\mathcal{X}$ and $\bigcap_{n\geq 1} \mathcal{Y}_n$. So

$$\sigma(\mathcal{S}) = \sigma\Big(\mathcal{X}, \bigcap_{n\geq 1} \mathcal{Y}_n\Big).$$

We show that $\mathcal{S}$ is independent of $\sigma(X_0)$. Since $(X_n)_{n\geq 1}$ is iid $\sigma(X_0)$ is independent of $\mathcal{X}$. We use this fact in what follows. Now, let $C \in \mathcal{S}$ and write $C = A \cap B$, where $A \in \mathcal{X}$ and $B \in \bigcap_{n\geq 1} \mathcal{Y}_n$. By (ii) we have $\mathrm{P}(B) = 0$ or $\mathrm{P}(B) = 1$. If $\mathrm{P}(B) = 0$ we have

$$0 = \mathrm{P}(S \cap C) = \mathrm{P}(S)\mathrm{P}(C).$$

If $\mathrm{P}(B) = 1$ we have

$$\mathrm{P}(S \cap C) = \mathrm{P}(S \cap A) = \mathrm{P}(S)\mathrm{P}(A) = \mathrm{P}(S)\mathrm{P}(C).$$

Finally, Dynkin's lemma completes the proof.

(iv) We have shown that

$$\sigma(X_0) \subset \bigcap_{n\geq 1} \sigma(\mathcal{X}, Y_n)$$

and that

$$\sigma(X_0) \text{ is indendent of } \sigma\Big(\mathcal{X}, \bigcap_{n\geq 1} \mathcal{Y}_n\Big).$$

Since $\sigma(X_0)$ is non-trivial, the two sigma algebras cannot be equal.