PROBABILITY THEORY (D-MATH)
EXERCISE SHEET 6

Exercise 1. Let $(X_n)_{n\geq 1}$ be iid random variables in L^2 such that X_1 has the same law as $-X_1$, $\mathrm{P}(X_1=0)>0$, and $X_1\in\mathbb{Z}$ a.s. For $n\geq 1$, define

$$S_n = X_1 + \cdots + X_n.$$

Show that there exists $c>0$ such that

$$\mathrm{P}(S_n=0) \underset{n\to\infty}{\sim} \frac{c}{\sqrt{n}}.$$

(For two sequences (a_n) and (b_n) of real numbers, we write $a_n \underset{n\to\infty}{\sim} b_n$ if $a_n/b_n \to 1$ as $n\to\infty$.)

Exercise 2. Let $(X_n)_{n\geq 1}$ be a an iid sequence of random variables with

$$\mathrm{P}(X_1=1)=\mathrm{P}(X_1=-1)=1/2.$$

(i) Show that there exists a constant $c>0$ such that for all $n\geq 1$ and positive real numbers $a_1,\ldots,a_n>0$ we have

$$\mathrm{P}(a_1X_1+\cdots+a_nX_n=0)\leq \frac{c}{\sqrt{n}}.$$

(ii) Show that there exists a constant $c>0$ such that for all $n\geq 1$ we have

$$\mathrm{P}(X_1+2X_2+\cdots+nX_n=0)\leq \frac{c}{n^{3/2}}.$$

Exercise 3 (The moment problem). [R]

In this exercise, we only consider random variables that are in L^p for all $p\geq 1$. We say that X is determined by its moments if for all random variables Y such that

$$\forall n\geq 1 \quad \mathrm{E}(X^n)=\mathrm{E}(Y^n), \tag{1}$$

we have $\mu_X=\mu_Y$.

(i) We first give an example of a random variable that is not determined by its moments. Let

$$X\sim e^Z \quad \text{where } Z\sim\mathcal{N}(0,1).$$

Let Y be a random variable taking values in $\{e^k : k\in\mathbb{Z}\}$ defined as follows:

$$\forall k\in\mathbb{Z}\quad \mathrm{P}(Y=e^k)=\frac{e^{-k^2/2}}{\Lambda} \quad \text{where } \Lambda=\sum_{k\in\mathbb{Z}} e^{-k^2/2}.$$

Show that

$$\forall n\geq 1\quad \mathrm{E}(X^n)=\mathrm{E}(Y^n)=e^{n^2/2}.$$

(ii) Let X be a random variable such that there exists $t>0$ such that $\mathrm{E}\big(e^{t|X|}\big)<\infty$. We show that then X is determined by its moments. First, check that $X\in L^p$ for all $p\geq 1$ and ϕ_X, the characteristic function of X, is infinitely differentiable on $\mathbb{R}$.

(iii) Fix $a \in \mathbb{R}$. Show that

$$\forall \epsilon \in (-t,t) \quad \phi_X(a+\epsilon) = \sum_{k=0}^{\infty} \frac{e^k}{k!} \phi_X^{(k)}(a).$$

(iv) Let ϕ_Y be the characteristic function of Y. Show that

$$\forall \epsilon \in (-t,t) \quad \phi_X(\epsilon) = \phi_Y(\epsilon).$$

(v) Show that $\phi_X(\epsilon) = \phi_Y(\epsilon)$ for all $\epsilon \in \mathbb{R}$. Conclude that $\mu_X = \mu_Y$.

Submission of solutions. Hand in your solutions by 18:00, 2/11/2024 following the instructions on the course website

`https://metaphor.ethz.ch/x/2024/hs/401-3601-00L/`

Note that only the exercises marked with [R] will be corrected.