PROBABILITY THEORY (D-MATH)
EXERCISE SHEET 13 – SOLUTION

Exercise 2. Let $(X_n)_{n\geq 1}$ be a an iid sequence of random variables with
$$\mathrm{P}(X_1 = 1) = \mathrm{P}(X_1 = -1) = 1/2.$$

(i) Show that there exists a constant $c > 0$ such that for all $n \geq 1$ and positive real numbers $a_1, \dots, a_n > 0$ we have
$$\mathrm{P}(a_1X_1 + \cdots + a_nX_n = 0) \leq \frac{c}{\sqrt{n}}.$$

(ii) Show that there exists a constant $c > 0$ such that for all $n \geq 1$ we have
$$\mathrm{P}(X_1 + 2X_2 + \cdots + nX_n = 0) \leq \frac{c}{n^{3/2}}.$$

Solution.

(1) This result requires a result outside the course called Sperner's theorem. (This exercise is not relevant for exam preparation.)

In this exercise, we identify elements of $\{-1, 1\}^n$ with subsets of $\{1, \dots, n\}$ by identifying $x \in \{-1, 1\}^n$ with $\{i : x_i = 1\}$.

Fix a positive integer n and positive real numbers $a_1, \dots, a_n$. Let
$$\mathcal{F} = \big\{(x_1, \dots, x_n) \in \{-1, 1\}^n : a_1x_1 + \cdots + a_nx_n = 0\big\}.$$
Now, suppose $x, y \in \mathcal{F}$ are such that x is a strict subset of y, then we have
$$0 = a_1x_1 + \cdots + a_nx_n < a_1y_1 + \cdots + a_ny_n,$$
which is a contradiction. Sperner's theorem (`https://en.wikipedia.org/wiki/Sperner%27s_theorem`) then implies that
$$|\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor}.$$
So we have
$$\mathrm{P}(a_1X_1 + \cdots + a_nX_n = 0) = |\mathcal{F}|/2^n \leq c/\sqrt{n},$$
for some universal constant $c > 0$.

(2) First, by independence the characteristic function of $X_1 + 2X_2 + \cdots nX_n$ is given by $\prod_{k=1}^n \cos(kt)$. Next, using section 5 of chapter 5 we have
$$\begin{aligned}
\mathrm{P}(X_1 + 2X_2 + \cdots + nX_n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \prod_{k=1}^n \cos(kt)\, dt \\
\big(\cos(t) = \cos(-t)\big) \quad &= \frac{1}{\pi}\int_0^{\pi} \prod_{k=1}^n \cos(kt)\, dt \\
&= \frac{1}{\pi}\int_0^{\pi/2} \prod_{k=1}^n \cos(kt)\, dt + \frac{1}{\pi}\int_{\pi/2}^{\pi} \prod_{k=1}^n \cos(kt)\, dt
\end{aligned}$$

Now, considering the change of variable $t \to \pi - t$ and using the fact that $\cos(k\pi - kt) = (-1)^k \cos(kt)$ we get

$$\frac{1}{\pi}\int_{\pi/2}^{\pi}\prod_{k=1}^{n}\cos(kt)\,dt = \frac{1}{\pi}\int_{0}^{\pi/2}\prod_{k=1}^{n}(-1)^k\cos(kt)\,dt.$$

So it is enough to show that

$$\frac{1}{\pi}\int_{0}^{\pi/2}\prod_{k=1}^{n}\cos(kt)\,dt \le cn^{-3/2}.$$

Now, we use the standard bound $\cos(\theta) \le 1 - \theta^2/4 \le \exp(-\theta^2/4)$ for $\theta \in [0, \pi/2]$. We obtain

$$\begin{aligned}
\int_{0}^{\pi/2}\prod_{k=1}^{n}\cos(kt)\,dt &\le \int_{0}^{\pi/2}\exp(-t^2\sum_{k=1}^{n}k^2/4)\,dt \\
(\sum_{k=1}^{n}k^2 \le n^3) \quad &\le \int_{0}^{\pi/2}\exp(-n^3t^2/4)\,dt \\
&= n^{-3/2}\int_{0}^{\pi/2}n^{3/2}\exp(-n^3t^2/4) \\
&\le cn^{-3/2},
\end{aligned}$$

where we used in the final inequality that $n^{3/2}\exp(n^3t^2/4)$ is the density of a $\mathcal{N}(0, 2n^{-3})$ random variable up to constants. This completes the proof.

Exercise 3 (The moment problem). [R]

In this exercise, we only consider random variables that are in L^p for all $p \geq 1$. We say that X is determined by its moments if for all random variables Y such that

$$\forall n \geq 1 \quad \mathrm{E}(X^n) = \mathrm{E}(Y^n), \tag{1}$$

we have $\mu_X = \mu_Y$.

(i) We first give an example of a random variable that is not determined by its moments. Let

$$X \sim e^Z \quad \text{where } Z \sim \mathcal{N}(0,1).$$

Let Y be a random variable taking values in $\{e^k : k \in \mathbb{Z}\}$ defined as follows:

$$\forall k \in \mathbb{Z} \quad \mathrm{P}(Y = e^k) = \frac{e^{-k^2/2}}{\Lambda} \quad \text{where } \Lambda = \sum_{k \in \mathbb{Z}} e^{-k^2/2}.$$

Show that

$$\forall n \geq 1 \quad \mathrm{E}(X^n) = \mathrm{E}(Y^n) = e^{n^2/2}.$$

(ii) Let X be a random variable such that there exists $t > 0$ such that $\mathrm{E}\big(e^{t|X|}\big) < \infty$. We show that then X is determined by its moments. First, check that $X \in L^p$ for all $p \geq 1$ and ϕ_X, the characteristic function of X, is infinitely differentiable on $\mathbb{R}$.

(iii) Fix $a \in \mathbb{R}$. Show that

$$\forall \epsilon \in (-t, t) \quad \phi_X(a + \epsilon) = \sum_{k=0}^{\infty} \frac{\epsilon^k}{k!} \phi_X^{(k)}(a).$$

(iv) Let ϕ_Y be the characteristic function of Y. Show that

$$\forall \epsilon \in (-t, t) \quad \phi_X(\epsilon) = \phi_Y(\epsilon).$$

(v) Show that $\phi_X(\epsilon) = \phi_Y(\epsilon)$ for all $\epsilon \in \mathbb{R}$. Conclude that $\mu_X = \mu_Y$.

Solution.

Let $n \geq 1$. Then we have

$$\begin{aligned}
\mathrm{E}(X^n) = \mathrm{E}(e^{nZ}) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{nz} e^{-z^2/2} \, dz \\
&= \frac{e^{n^2/2}}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-(z-n)^2/2} \, dz \\
&= \frac{e^{n^2/2}}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-z^2/2} \, dz \\
&= e^{n^2/2}.
\end{aligned}$$

We also have

$$
\begin{aligned}
\mathrm{E}(Y^n) &= \frac{1}{\Lambda}\sum_{k\in\mathbb{Z}} e^{kn}e^{-k^2/2} \\
&= \frac{e^{n^2/2}}{\Lambda}\sum_{k\in\mathbb{Z}} e^{-(k-n)^2/2} \\
&= \frac{e^{n^2/2}}{\Lambda}\sum_{k\in\mathbb{Z}} e^{-k^2/2} \\
&= e^{n^2/2}.
\end{aligned}
$$

(i) X is in L^p for every $p \geq 1$ because for every p, $|x|^p \leq e^{t|x|}$ for $|x|$ large enough. The fact that ϕ_X is infinitely differntiable is a consequence of the theorem in section 5 of chapter 5. We also use the following fact later:

$$\mathrm{E}(e^{t|Y|/2}) < \infty.$$

To show this first note that $\mathrm{E}(|Y|^n) = \mathrm{E}(|X|^n)$ for even n and and $\mathrm{E}(|Y|^n) \leq \mathrm{E}(|Y|^{n+1}) = \mathrm{E}(|X|^{n+1})$ for odd n. Second, observe that

$$
\begin{aligned}
\mathrm{E}(e^{t|Y|/2}) &= \mathrm{E}\Big(\sum_{k\geq 0}(t/2)^k|Y|^k/k!\Big) \\
&= \sum_{k\geq 0}(t/2)^k\mathrm{E}(|Y|^k)/k! \\
&\leq \sum_{k\geq 0}(t/2)^{2k}\mathrm{E}(|X|^{2k})/(2k)! + \sum_{k\geq 0}(t/2)^{2k+1}\mathrm{E}(|X|^{2k+2})/(2k+1)!,
\end{aligned}
$$

which is finite because $\mathrm{E}(e^{t|X|}) < \infty$. In what follows, we may assume that $t > 0$ is such that

$$\mathrm{E}(e^{t|Y|}), \mathrm{E}(e^{t|X|}) < \infty.$$

(ii) Let $a \in \mathbb{R}$ and let $\epsilon \in (-t, t)$. For $n \geq 1$, using section 5 of chapter 5 we have

$$
\begin{aligned}
&|\phi_X(a+\epsilon) - \sum_{k=0}^{n}\frac{\epsilon^k}{k!}\phi_X^{(k)}(a)| \\
&\qquad = |\mathrm{E}\big(e^{i(a+\epsilon)X} - \sum_{k=0}^{n}\frac{\epsilon^k}{k!}(iX)^k e^{iaX}\big)| \\
&\qquad \leq \mathrm{E}\Big(|e^{iaX}||e^{i\epsilon X} - \sum_{k=0}^{n}\frac{\epsilon^k}{k!}(iX)^k|\Big),
\end{aligned}
$$

which converges to 0 as $n \to \infty$, as $\mathrm{E}(e^{i\epsilon|X|}) < \infty$.

(iii) We note that at an analogous formula to the one in the previous part holds for Y by the same argument. Now, since

$$\phi_X^{(k)}(0) = \mathrm{E}((iX)^k) = \mathrm{E}((iY)^k) = \phi_Y^{(k)}(0),$$

the result follows from the formula in the previous part.

(iv) We have that $\phi_X(\epsilon) = \phi_Y(\epsilon)$ for all $\epsilon \in (-t, t)$. This implies that for all $k \geq 0$, $\phi_X^{(k)}(t) = \phi_Y^{(k)}(t)$ and $\phi_X^{(k)}(-t) = \phi_Y^{(k)}(-t)$. Applying the argument in the previous part, we then get that $\phi_X(\epsilon) = \phi_Y(\epsilon)$ for all $\epsilon \in (-2t, 2t)$. Continuing like this we inductively obtain that $\phi_X = \phi_Y$, so $\mu_X = \mu_Y$, as desired.

Exercise 1. Let $(X_n)_{n\geq 1}$ be iid random variables in L^2 such that X_1 has the same law as $-X_1$, $\mathrm{P}(X_1 = 0) > 0$, and $X_1 \in \mathbb{Z}$ a.s. For $n \geq 1$, define

$$S_n = X_1 + \cdots + X_n.$$

Show that there exists $c > 0$ such that

$$\mathrm{P}(S_n = 0) \underset{n\to\infty}{\sim} \frac{c}{\sqrt{n}}.$$

(For two sequences (a_n) and (b_n) of real numbers, we write $a_n \underset{n\to\infty}{\sim} b_n$ if $a_n/b_n \to 1$ as $n \to \infty$.)

Solution.

To be added. (The solution we had in mind had a mistake.)