PROBABILITY THEORY (D-MATH)
EXERCISE SHEET 6

Exercise 1. Let (E, d) and (E', d') be metric spaces and let $f : E \to E'$ be a continuous function. Let $(X_n)_{n\geq 1}, X$ be random variables taking values in E such that

$$X_n \xrightarrow{(d)} X.$$

Show that

$$f(X_n) \xrightarrow{(d)} f(X).$$

Exercise 2. [R] Let $p \in (0, 1)$ and let $(X_n)_{n\geq 1}$ be a sequence of random variables where $X_n \sim \text{Geo}(p/n)$. Show that X_n/n converges in distribution to a random variable Y. What is the distribution of Y?

Exercise 3. [R] Let $(X_n)_{n\geq 1}$ be a sequence of real-valued random variables where X_n has density p_n (with respect to Lebesgue measure Leb). Suppose there is a measurable function such that for Leb-almost all $x \in \mathbb{R}$ we have

$$p_n(x) \to p(x) \text{ as } n \to \infty.$$

(i) Is p always the density of some random variable? Justify your answer.

(ii) Assume that there is an integrable measurable function (with respect to Leb)

$$q : \mathbb{R} \to \mathbb{R}_{\geq 0}$$

such that for all $n \geq 1$ and Leb-almost all x we have

$$p_n(x) \leq q(x).$$

Then show that p is the density of some random variable X and that X_n converges in distribution to X.

Exercise 4. Let $(X_n)_{n\geq 1}$ be a sequence of real-valued random variables converging in distribution to a uniformly distributed random variable in $[0, 1]$. Let $(Y_n)_{n\geq 1}$ be a sequence of real-valued random variables converging in probability to 0. Show that

$$\mathrm{P}(X_n < Y_n) \to 0 \text{ as } n \to \infty.$$

Submission of solutions. Hand in your solutions by 18:00, 8/11/2024 following the instructions on the course website

`https://metaphor.ethz.ch/x/2024/hs/401-3601-00L/`

Note that only the exercises marked with [R] will be corrected.