PROBABILITY THEORY (D-MATH)
EXERCISE SHEET 7 – SOLUTION

Exercise 1. Let (E, d) and (E', d') be metric spaces and let $f : E \to E'$ be a continuous function. Let $(X_n)_{n\geq 1}, X$ be random variables taking values in E such that

$$X_n \xrightarrow{(d)} X.$$

Show that

$$f(X_n) \xrightarrow{(d)} f(X).$$

Solution. Let $g : E' \to \mathbb{R}$ be continuous and bounded. Then $g \circ f : E \to \mathbb{R}$ is continuous and bounded and because $X_n \xrightarrow{(d)} X$ we have

$$\mathrm{E}(g(f(X_n))] \to \mathrm{E}[g(f(X))],$$

which proves that $f(X_n) \xrightarrow{(d)} f(X)$.

Exercise 2. [R] Let $p \in (0,1)$ and let $(X_n)_{n\geq 1}$ be a sequence of random variables where $X_n \sim \mathrm{Geo}(p/n)$. Show that X_n/n converges in distribution to a random variable Y. What is the distribution of Y?

Solution. We claim that $X_n/n \xrightarrow{(d)} Y$, where $Y \sim \mathrm{Exp}(1)$. To show this, we use the characterisation of convergence in distribution for real random variables, that is, we show that for all $t \in \mathbb{R}$,

$$F_{X_n/n}(t) \to F_Y(t).$$

For $t \leq 0$, $0 = F_{X_n/n}(t) = F_Y(t)$ for all n so the convergence is true trivially. Let $t > 0$. Then

$$F_{X_n/n}(t) = \mathrm{P}(X_n \leq tn) = 1 - \mathrm{P}(X_n > tn) = 1 - (1 - p/n)^{\lfloor tn \rfloor} \to 1 - e^{-t},$$

which is equal to $F_Y(t)$. This completes the proof.

Exercise 3. [R] Let $(X_n)_{n\geq 1}$ be a sequence of real-valued random variables where X_n has density p_n (with respect to Lebesgue measure Leb). Suppose there is a measurable function such that for Leb-almost all $x \in \mathbb{R}$ we have

$$p_n(x) \to p(x) \text{ as } n \to \infty.$$

(i) Is p always the density of some random variable? Justify your answer.

(ii) Assume that there is an integrable measurable function (with respect to Leb)

$$q : \mathbb{R} \to \mathbb{R}_{\geq 0}$$

such that for all $n \geq 1$ and Leb-almost all x we have

$$p_n(x) \leq q(x).$$

Then show that p is the density of some random variable X and that X_n converges in distribution to X.

Solution.

(i) No, for instance consider $X_n \sim \text{Unif}[0, 1/n]$. Then X_n has density $n1_{[0,1/n]}$ which converges almost everywhere to the constant 0 function, but 0 is not a density.

(ii) First, we show that p is a density. Being a almost everywhere pointwise limit of non-negative functions it is non-negative and furthermore by dominated convergence we have

$$\int_{\mathbb{R}} p(x)dx = \lim_{n\to\infty} \int_{\mathbb{R}} p_n(x)dx = 1,$$

showing that p is a density of some random varible X. Furthermore, for all $t \in \mathbb{R}$, again by dominated convergence, we have

$$F_X(t) = \mathrm{P}(X \leq t) = \int_{-\infty}^{t} p(x)dx = \lim_{n\to\infty} \int_{-\infty}^{t} p_n(x)dx = \lim_{n\to\infty} F_{X_n}(t),$$

which shows that $X_n \xrightarrow{(d)} X$ (using the characterisation of convergence in distribution on $\mathbb{R}$).

Exercise 4. Let $(X_n)_{n\geq 1}$ be a sequence of real-valued random variables converging in distribution to a uniformly distributed random variable in $[0,1]$. Let $(Y_n)_{n\geq 1}$ be a sequence of real-valued random variables converging in probability to 0. Show that

$$\mathrm{P}(X_n < Y_n) \to 0 \text{ as } n \to \infty.$$

Solution. Let $\epsilon > 0$

$$\begin{aligned}
\mathrm{P}(X_n < Y_n) &= \mathrm{P}(\{X_n < Y_n, |Y_n| > \epsilon\} \cup \{X_n < Y_n, |Y_n| \leq \epsilon\}) \\
\text{(union bound)} \quad &\leq \mathrm{P}(X_n < Y_n, |Y_n| > \epsilon) + \mathrm{P}(X_n < Y_n, |Y_n| \leq \epsilon) \\
&\leq \mathrm{P}(|Y_n| > \epsilon) + \mathrm{P}(X_n \leq \epsilon) \\
&\longrightarrow 0 + \epsilon,
\end{aligned}$$

where we used that $Y_n \xrightarrow{(\mathrm{P})} 0$ and $X_n \xrightarrow{(d)} \mathrm{Unif}[0,1]$ in the last line. Since ϵ was arbitrary, this completes the proof.