

**PROBABILITY THEORY (D-MATH)
EXERCISE SHEET 8**

Exercise 1. [R] Let $(X_n)_{n \geq 1}$ be an iid sequence of $\mathcal{N}(0, 1)$ random variables. For $n \geq 1$, define

$$Y_n = \frac{1}{n} \sum_{k=1}^n \sqrt{k} X_k.$$

Does Y_n converge in distribution? What is the limit?

Exercise 2. [R] Let $(X_n)_{n \geq 1}$ be a sequence of iid $\mathcal{U}[0, 1]$ random variables.

- (i) Show that $n \min(X_1, \dots, X_n)$ converges in distribution to a random variable Y . What is the distribution of Y ?
- (ii) Show that

$$(X_1 + \dots + X_n) \min(X_1, \dots, X_n) \xrightarrow{(d)} Y/2.$$

Exercise 3 (Normality of the t-statistic). [R] Let $(X_n)_{n \geq 1}$ be iid real-valued random variables in L^2 . Let $m = \mathbb{E}(X_1)$ and $\sigma^2 = \text{Var}(X_1)$. For $n \geq 1$, define

$$\bar{X}_n = \frac{X_1 + \dots + X_n}{n} \quad \text{and} \quad S_n^2 = \frac{1}{n-1} \sum_{k=1}^n (X_k - \bar{X}_n)^2.$$

The aim of this exercise is to show that

$$\frac{X_1 + \dots + X_n - nm}{\sqrt{nS_n^2}} \xrightarrow{(d)} Z, \quad \text{where } Z \sim \mathcal{N}(0, 1). \quad (1)$$

- (i) Show that $S_n^2 \rightarrow \sigma^2$ a.s.
- (ii) Show that $\frac{X_1 + \dots + X_n}{\sqrt{n\sigma^2}} \xrightarrow{(d)} Z$, where $Z \sim \mathcal{N}(0, 1)$.
- (iii) Prove (1).

Exercise 4 (Skorokhod representation on the reals). Let $(X_n)_{n \geq 1}, X$ be real-valued random variables such that $X_n \xrightarrow{(d)} X$. The aim of this is to construct a probability space carrying these random variables such that $X_n \xrightarrow{a.s.} X$. For a distribution function F , we define

$$F^{-1} : (0, 1) \rightarrow \mathbb{R}, \quad \text{by } F^{-1}(t) = \inf\{s : F(s) > t\}.$$

Let (F_n) and F be the distribution functions of (X_n) and (X) , and let $U \sim \mathcal{U}(0, 1)$.

- (i) Show that $F_n^{-1}(U)$ has the same distribution as X_n for all $n \geq 1$ and that $F^{-1}(U)$ has the same distribution as X .
- (ii) Show that

$$F_n^{-1}(U) \xrightarrow{a.s.} F^{-1}(U) \quad \text{as } n \rightarrow \infty.$$

Submission of solutions. Hand in your solutions by 18:00, 16/11/2024 following the instructions on the course website

<https://metaphor.ethz.ch/x/2024/hs/401-3601-00L/>

Note that only the exercises marked with [R] will be corrected.