PROBABILITY THEORY (D-MATH)
EXERCISE SHEET 8 – SOLUTION

Exercise 1. [R] Let $(X_n)_{n\geq 1}$ be an iid sequence of $\mathcal{N}(0,1)$ random variables. For $n \geq 1$, define

$$Y_n = \frac{1}{n}\sum_{k=1}^{n} \sqrt{k} X_k.$$

Does Y_n converge in distribution? What is the limit?

Solution. Throughout this exercise we use the fact if $Z \sim \mathcal{N}(0,\sigma^2)$, then

$$\phi_Z(t) = \exp(-\sigma^2 t^2/2).$$

We show that Y_n converges in distribution to a $\mathcal{N}(0,1/2)$ random variable. By the Levy's theorem in section 4 of chapter 7 it is enough to show that

$$\forall t \in \mathbb{R} \quad \phi_{Y_n}(t) \to \exp(-t^2/4).$$

Let $t \in \mathbb{R}$. Note that

$$\phi_{\sqrt{k}X_k/n}(t) = \exp\left(-\frac{kt^2}{2n^2}\right),$$

so by independence of (X_k) we have

$$\phi_{Y_n}(t) = \exp\left(-\frac{t^2\sum_{k=1}^n k}{2n^2}\right) = \exp\left(-\frac{t^2 n(n+1)}{4n^2}\right) \to \exp(-t^2/4),$$

as required.

Exercise 2. [R] Let $(X_n)_{n\geq 1}$ be a sequence of iif $\mathcal{U}[0,1]$ random variables.

(i) Show that $n\min(X_1,\dots,X_n)$ converges in distribution to a random variable Y. What is the distribution of Y?

(ii) Show that

$$(X_1+\cdots+X_n)\min(X_1,\dots,X_n) \xrightarrow{(d)} Y/2.$$

Solution.

(i) We show that $Y_n = n\min(X_1,\dots,X_n)$ converges in distribution to a $Y \sim \mathrm{Exp}(1)$ random variable. It is enough to show that for all $t \in \mathbb{R}$ we have

$$\mathrm{P}(Y_n \geq t) \to f(t),$$

where $f(t) = e^{-t}$ for $t > 0$ and $f(t) = 1$ for $t \leq 0$. When $t \leq 0$, $\mathrm{P}(Y_n \geq t) = 1$ for all n so the above convergence holds trivially. Let $t > 0$. Then for n large enough we have

$$\begin{aligned}
\mathrm{P}(Y_n \geq t) &= \mathrm{P}(\min(X_1,\dots,X_n) \geq t/n) \\
&= \mathrm{P}(\cap_{k=1}^n \{X_k > t/n\}) \\
(X_k \text{ are iid}) \quad &= \mathrm{P}(X_1 > t/n)^n \\
(X_1 \sim \mathcal{U}[0,1]) \quad &= (1-t/n)^n,
\end{aligned}$$

which converges to e^{-t}, as desired.

(ii) Note that $S_n = (X_1+\cdots+X_n)/n \xrightarrow{\mathrm{P}} 1/2$ by the weak law of large numbers. Therefore, this convergence also holds in distribution. By Slutsky's theorem $(S_n, Y_n) \xrightarrow{(d)} (1/2, Y)$. Since $g : \mathbb{R}^2 \to \mathbb{R}$, given by $(x,y) \mapsto xy$ is continuous, we have $S_nY_n \xrightarrow{(d)} Y/2$, completing the proof.

Exercise 3 (Normality of the t-statistic). [R] Let $(X_n)_{n\geq 1}$ be iid real-valued random variables in L^2. Let $m = \mathrm{E}(X_1)$ and $\sigma^2 = \mathrm{Var}(X_1$. For $n \geq 1$, define

$$\bar{X}_n = \frac{X_1 + \cdot + X_n}{n} \quad \text{and} \quad S_n^2 = \frac{1}{n-1}\sum_{k=1}^{n}(X_k - \bar{X}_n)^2.$$

The aim of this exercise is to show that

$$\frac{X_1 + \cdots + X_n - nm}{\sqrt{nS_n^2}} \xrightarrow{(d)} Z, \text{ where } Z \sim \mathcal{N}(0,1). \tag{1}$$

(i) Show that $S_n^2 \to \sigma^2$ a.s.

(ii) Show that $\frac{X_1+\cdots+X_n}{\sqrt{n\sigma^2}} \xrightarrow{(d)} Z$, where $Z \sim \mathcal{N}(0,1)$.

(iii) Prove (1).

Solution. Refer to section 8 of chapter 7 for a solution.

Exercise 4 (Skorokhod representation on the reals). Let $(X_n)_{n\geq 1}, X$ be real-valued random variables such that $X_n \xrightarrow{(d)} X$. The aim of this is to construct a probability space carrying these random variables such that $X_n \xrightarrow{a.s.} X$. For a distribution function F, we define

$$F^{-1} : (0,1) \to \mathbb{R}, \quad \text{by } F^{-1}(t) = \inf\{s : F(s) > t\}.$$

Let (F_n) and F be the distribution functions of (X_n) and (X), and let $U \sim \mathcal{U}(0,1)$.

(i) Show that $F_n^{-1}(U)$ has the same distribution as X_n for all $n \geq 1$ and that $F^{-1}(U)$ has the same distribution as X.

(ii) Show that

$$F_n^{-1}(U) \xrightarrow{a.s.} F^{-1}(U) \quad \text{as } n \to \infty.$$

Solution.

(i) Let $t \in \mathbb{R}$. We claim that $\mathrm{P}(F^{-1}(U) \leq t) = F(t)$. First, note by the definition of F^{-1} we have that

$$\{U < F(t) \subset \{F^{-1}(U) \leq t\}.$$

Second, since F is right-continuous, so is F^{-1} and so we have

$$\{F^{-1}U \leq t\} \subset \{U \leq F(t)\}.$$

Since $\mathrm{P}(U = F(t)) = 0$, the above two inclusions imply the claim, completing the proof. The proofs for the X_n's are analogous.

(ii) Note that F^{-1} is right-continuous and non-decreasing, so it has at most countably many discontinuity point. Let $u \in (0,1)$ be a continuity point. Let $x > F^{-1}(u)$ be a continuity point of F. Then we have $F(x) > u$ and by the definition of convergence in distribution we have that $F_n(x) \to F(x)$, so $F_n(x) > u$ for large n. Taking a sequence of continuity points x of F decreasing to u we conclude that

$$\limsup F_n^{-1}(u) \leq F^{-1}(u).$$

Fix $\epsilon > 0$. We claim that there exists $x \geq F^{-1} - \epsilon$, such that $F(x) < u$. Indeed, if not, there exists δ such that $F(x) \geq u$ for all $x \in (F^{-1}(u) - \delta, x)$, and so $F^{-1}(u') \leq F^{-1}(u) - \delta$ for all $u' < u$, violating left continuity of F^{-1} at u. Moreover, since continuity points of F are dense, we may assume that x is a continuity point of F. So, $F_n(x) < u$ for large n as well. So $F_n^{-1}(u) \geq x$. Taking $\epsilon \to 0$ we conclude that

$$\liminf F_n^{-1}(u) \geq F^{-1}(u).$$

So we have shown that for all continuity points u of F^{-1} we have that $F_n^{-1}(u) \to F^{-1}(u)$. Since the set of discontinuity points is at most countable, U is almost surely a continuity point, and therefore almost surely $F_n^{-1}(U) \to F^{-1}(U)$, as required.