Prof. Vincent Tassion HS 2024

PROBABILITY THEORY (D-MATH)
EXERCISE SHEET 8

Exercise 1. [R] Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a probability space and let $\mathcal{A} = \{\Omega_1, \Omega_2, \dots, \Omega_n\}$ be a partition of Ω. Let X be a real-valued $\sigma(\mathcal{A})$-measurable random variable. Show that there exist real numbers $\lambda_1, \dots, \lambda_n$ such that

$$X = \sum_{i=1}^{n} \lambda_i 1_{\Omega_i}.$$

Exercise 2. [R] Fix $n \geq 1$. Let $X \sim \mathrm{Unif}[0,1]$ and let $Y = \lfloor n \cdot X \rfloor$. Compute $\mathrm{E}(X|Y)$.

Exercise 3. Fix $n \geq 2$. Let X, Y be two numbers chosen uniformly at random from $\{1, 2, \dots, n\}$ without replacement. Define the event $A = \{Y > X\}$.

(i) Compute $\mathrm{E}(Y|A)$.
(ii) Compute $\mathrm{E}\big(\max(X,Y) | \min(X,Y)\big)$.

Exercise 4. Let X, Y be real-valued random variables taking finitely many values. Define the random variable

$$\mathrm{Var}(X|Y) = \mathrm{E}(X^2|Y) - \mathrm{E}(X|Y)^2.$$

Show that

$$\mathrm{Var}(X) = \mathrm{E}\big(\mathrm{Var}(X|Y)\big) + \mathrm{Var}\big(\mathrm{E}(X|Y)\big).$$

Exercise 5. Let $(X_n)_{n\geq 1}, (Y_n)_{n\geq 1}, X, Y$ be random variables. Assume that for all $n \geq 1$, X_n and Y_n are independent, and that X and Y are independent. Suppose

$$X_n \xrightarrow{(d)} X \quad \text{and} \quad Y_n \xrightarrow{(d)} Y.$$

Then show that

$$(X_n, Y_n) \xrightarrow{(d)} (X, Y).$$

Submission of solutions. Hand in your solutions by 18:00, 22/11/2024 following the instructions on the course website

`https://metaphor.ethz.ch/x/2024/hs/401-3601-00L/`

Note that only the exercises marked with [R] will be corrected.