PROBABILITY THEORY (D-MATH)
EXERCISE SHEET 8 – SOLUTION

Exercise 1. [R] Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a probability space and let $\mathcal{A} = \{\Omega_1, \Omega_2, \ldots, \Omega_n\}$ be a partition of Ω. Let X be a real-valued $\sigma(\mathcal{A})$-measurable random variable. Show that there exist real numbers $\lambda_1, \ldots, \lambda_n$ such that

$$X = \sum_{i=1}^{n} \lambda_i 1_{\Omega_i}.$$

Solution.

First note that

$$\sigma(A) = \left\{ \bigcup_{i \in A} \Omega_i : A \subset [n] \right\}.$$

Indeed, every element of the above set is a union of finitely many sets in A and is a sigma-algebra containing A. We observe that since the Ω_i's are disjoint, for all $i \in [n]$ and $S \in \sigma(A)$, either $\Omega_i \cap S = \emptyset$ or $\Omega_i \subset S$.

Now, for all $i \in [n]$ we show that X is constant on Ω_i. Suppose not. Then there exists $\omega, \omega' \in \Omega_i$ such that $X(\omega) \neq X(\omega')$. Then $S = X^{-1}(X(\omega)) \cap \Omega_i \neq \emptyset$ but $X^{-1}(X(\omega))$ does not contain Ω_i. Then by our description of $\sigma(A)$, S is not contained in $\sigma(A)$, which contradicts the observation above. So X is constant on each Ω_i. Let λ_i be the value X takes on Ω_i. Therefore, $X = \sum_{i=1}^{n} \lambda_i 1_{\Omega_i}$, as required.

Exercise 2. [R] Fix $n \geq 1$. Let $X \sim \mathrm{Unif}[0,1]$ and let $Y = \lfloor n \cdot X \rfloor$. Compute $\mathrm{E}(X|Y)$.

Solution. We use the definition of conditional expectation for discrete random variables (see chapter 9). Note that Y belongs to $S = \{0, \ldots, n-1\}$ almost surely and for each $k \in S$,

$$\mathrm{P}(Y = k) = \mathrm{P}(X \in [k/n, (k+1)/n)) = 1/n.$$

Let $k \in S$. Then

$$\begin{aligned}
\mathrm{E}(X|Y = k) &= \frac{\mathrm{E}(X 1_{Y=k})}{\mathrm{P}(Y = k)} \\
&= n\mathrm{E}(X 1_{X \in [k/n,(k+1)/n)]}) \\
&= n \int_{k/n}^{(k+1)/n} x \, dx \\
&= k/n + 1/(2n).
\end{aligned}$$

So

$$\mathrm{E}(X|Y) = Y/n + 1/(2n).$$

Exercise 3. Fix $n \geq 2$. Let X, Y be two numbers chosen uniformly at random from $\{1, 2, \ldots, n\}$ without replacement. Define the event $A = \{Y > X\}$.

(i) Compute $\mathrm{E}(Y|A)$.

(ii) Compute $\mathrm{E}\big(\max(X,Y)|\min(X,Y)\big)$.

Solution.

Formally, the definition of (X, Y) is a uniform element of $\{(i,j) \in [n]^2 : i \neq j\}$.

(i) First, note that $\mathrm{P}(A) = 1/2$, for instance because (X, Y) has the same distribution as (Y, X). We get

$$\begin{aligned}
\mathrm{E}(Y|A) &= \frac{\mathrm{E}(Y 1_A)}{\mathrm{P}(A)} \\
&= \frac{2}{n(n-1)} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} j \\
&= \frac{2}{n(n-1)} \sum_{i=1}^{n-1} (n+i+1)(n-i)/2 \\
&= \frac{1}{n(n-1)} \sum_{i=1}^{n-1} n^2 + n - i^2 - i \\
&= \frac{1}{n(n-1)} \big(n^2(n-1) + n(n-1) - (n-1)n(2n-1)/6 - n(n-1)/2\big) \\
&= n + 1 - (2n-1)/6 - 1/2 \\
&= (2n+2)/3.
\end{aligned}$$

Remark. Note that $\mathrm{E}(Y) \sim n/2$ but after conditioning that Y is bigger than X, the expactation goes all the way up to $\sim 2n/3$.

(ii) Note that $\min(X, Y)$ takes values in $\{1, \ldots, n-1\}$. Fix $k \in \{0, \ldots, n-1\}$. Note that

$$P(\min(X, Y) = k) = \frac{|\{(k, i) : k+1 \leq i \leq n\}| \cup |\{(k, i) : k+1 \leq i \leq n\}|}{n(n-1)} = 2(n-k)/(n(n-1)).$$

In the following calculation we will use the fact that for $i \in \{k+1, \ldots, n\}$

$$P(\max(X, Y) = i, \min(X, Y) = k) = \frac{2}{n(n-1)}.$$

We get

$$\begin{aligned} E(\max(X, Y) | \min(X, Y)) &= \frac{n(n-1)}{2(n-k)} \sum_{i=k+1}^{n} 2i/n(n-1) \\ &= (n+k+1)/2. \end{aligned}$$

Therefore,

$$E(\max(X, Y) | \min(X, Y)) = (\min(X, Y) + n + 1)/2.$$

Exercise 4. Let X, Y be real-valued random variables taking finitely many values. Define the random variable

$$\operatorname{Var}(X|Y) = \mathrm{E}(X^2|Y) - \mathrm{E}(X|Y)^2.$$

Show that

$$\operatorname{Var}(X) = \mathrm{E}\big(\operatorname{Var}(X|Y)\big) + \operatorname{Var}\big(\mathrm{E}(X|Y)\big).$$

Solution. Since X takes finitely many values, it is bounded, so it is in L^2. Therefore, all the quantities in the exercise are well-defined. We have

$$\begin{aligned}
\mathrm{E}\big(\operatorname{Var}(X|Y)\big) + \operatorname{Var}\big(\mathrm{E}(X|Y)\big) &= \mathrm{E}\big(\mathrm{E}(X^2|Y) - \mathrm{E}(X|Y)^2\big) + \mathrm{E}\big(\mathrm{E}(X|Y)^2\big) - \mathrm{E}\big(\mathrm{E}(X|Y)\big)^2 \\
&= \mathrm{E}\big(\mathrm{E}(X^2|Y)\big) - \mathrm{E}\big(-\mathrm{E}(X|Y)^2\big) + \mathrm{E}\big(\mathrm{E}(X|Y)^2\big) - \mathrm{E}(X)^2 \\
&= \mathrm{E}(X^2) - \mathrm{E}(X)^2 \\
&= \operatorname{Var}(X).
\end{aligned}$$

Remark. It was not important that X, Y take finitely many values, we just needed that X is in L^2.

Exercise 5. Let $(X_n)_{n\geq 1}, (Y_n)_{n\geq 1}, X, Y$ be real-valued random variables. Assume that for all $n \geq 1$, X_n and Y_n are independent, and that X and Y are independent. Suppose

$$X_n \xrightarrow{(d)} X \quad \text{and} \quad Y_n \xrightarrow{(d)} Y.$$

Then show that

$$(X_n, Y_n) \xrightarrow{(d)} (X, Y).$$

Solution.

We use the characterisation for convergence in distribution using characteristic functions. Since $X_n \xrightarrow{(d)} X$ and $Y_n \xrightarrow{(d)} Y$ we have that

$$\forall t \in \mathbb{R} \quad \phi_{X_n}(t) \to \phi_X(t) \quad \text{and} \quad \phi_{Y_n}(t) \to \phi_Y(t).$$

By independence, we have

$$\forall (s,t) \in \mathbb{R}^2 \quad \phi_{(X_n,Y_n)}(s,t) = \phi_{X_n}(s)\phi_{Y_n}(t) \quad \text{and} \quad \phi_{(X,Y)}(s,t) = \phi_X(s)\phi_Y(t).$$

So we get for all $(s,t) \in \mathbb{R}^2$ that

$$\phi_{(X_n,Y_n)}(s,t) \to \phi_{(X,Y)}(s,t),$$

which completes the proof.