CHAPTER 11

CONDITIONAL EXPECTATION III : PROPERTIES.

<u>Goals:</u>
- presentation of the properties of conditional expectation
- specific properties : "effect" of the σ-algebra
- properties of an expectation.

<u>Set up.</u>
- $(\Omega, \mathcal{F}, P)$ probability space.
- $L^1 = \{ X : \Omega \to \mathbb{R}$ r.v. $E(|X|) < \infty \}$.

1 BASIC PROPERTIES

<u>Prop:</u> Let $\mathcal{G} \subset \mathcal{F}$ σ-algebra. $X \in L^1$.

① If X is $\mathcal{G}$-measurable, then
$$E(X \mid \mathcal{G}) = X \quad \text{a.s.}$$

② If X is independent of $\mathcal{G}$, then
$$E(X \mid \mathcal{G}) = E(X) \quad \text{a.s.}$$

③ $E(E(X \mid \mathcal{G})) = E(X)$.

Proof. (1) Since we trivially have $E(1_A X) = E(1_A X)$ for every $A \in \mathcal{G}$, if X is $\mathcal{G}$-measurable, it is a version of $E(X|\mathcal{G})$.

(2) $E(X)$ is constant, hence it is $\mathcal{G}$-measurable.

Furthermore $\forall A \in \mathcal{G}$, $E(X 1_A) \overset{\text{indep.}}{=} E(X) E(1_A)$

$$= E\left(E(X) 1_A\right).$$

(3) By applying (P2) with $A = \Omega \in \mathcal{G}$, we get

$$E(X.1) = E\left(E(X|\mathcal{G}).1\right). \qquad \square$$

Application: • If $X \in L^1$, we have

$$E(X|X) = X \text{ a.s. by (1)}$$

$$E(X|\{\Omega, \emptyset\}) = E(X) \text{ a.s. by (2)}$$

• If X, Y indep. $E(X|Y) = E(X)$ a.s.

2 LINEARITY

<u>Prop.</u> Let $X, Y \in L^1$ $a, b \in \mathbb{R}$, or $X, Y \geq 0$ $a, b \geq 0$

$$E(aX + bY \mid \mathcal{G}) = a E(X \mid \mathcal{G}) + b E(Y \mid \mathcal{G}). \quad \text{a.s.}$$

<u>Proof.</u> <u>L^1 case</u>

Let Z $\mathcal{G}$-measurable bounded.

$$E\big(Z(aX+bY)\big) = a\, E(ZX) + b\, E(ZY).$$

$$= a\, E\big(Z E(X \mid \mathcal{G})\big) + b\, E\big(Z E(Y \mid \mathcal{G})\big)$$

$$= E\Big(Z\big(a E(X \mid \mathcal{G}) + b E(Y \mid \mathcal{G})\big)\Big).$$

Since $a\, E(X \mid \mathcal{G}) + b\, E(Y \mid \mathcal{G})$ is $\mathcal{G}$-meas., this concludes the proof.

<u>≥ 0 case</u> : same proof with $Z \geq 0$.

3 MONOTONICITY.

<u>Rk:</u> <u>positivity of the integral</u>. If $X \geq 0$ and $X \in L^1$ we gave two definitions of $E(X \mid \mathcal{G})$. These two definitions coincide by uniqueness. We always have

$$X \geq 0 \text{ a.s.} \Rightarrow E(X \mid \mathcal{G}) \geq 0 \quad \text{a.s.}$$

Prop. Let $X, Y \in L^1$ & $X, Y \geq 0$

$$X \leq Y \text{ a.s.} \implies E(X|\mathcal{G}) \leq E(Y|\mathcal{G}) \text{ a.s.}$$

Pf. $Y = X + \underbrace{Y-X}_{\geq 0 \text{ a.s.}}$. By linearity, we have

$$E(Y|\mathcal{G}) = E(X|\mathcal{G}) + \underbrace{E(Y-X|\mathcal{G})}_{\geq 0} \text{ a.s.} \qquad \blacksquare$$

4 TOWER PROPERTY.

Prop. Let $\mathcal{G}, \mathcal{H} \subset \mathcal{F}$ σ-algebras s.t. $\mathcal{H} \subset \mathcal{G}$.

- $E\big(E(X|\mathcal{H}) \,|\, \mathcal{G}\big) = E(X|\mathcal{H})$ a.s.
- $E\big(E(X|\mathcal{G}) \,|\, \mathcal{H}\big) = E(X|\mathcal{H})$ a.s.

Proof. • For the first equality, notice that $E(X|\mathcal{H})$ is $\mathcal{G}$-measurable.

• For the second equality, let $X' = E(X|\mathcal{G})$.

For every Z $\mathcal{H}$-measurable bounded, we have

$$E(Z\,E(X'|\mathcal{H})) = E(ZX')$$
$$= E(Z\,E(X|\mathcal{G})) \underset{\substack{\uparrow \\ \text{because } Z \text{ is } \mathcal{G}\text{-meas.}}}{=} E(ZX)$$

Since $E(X'|\mathcal{H})$ is $\mathcal{H}$-measurable, we have

$$E(X'|\mathcal{H}) = E(X|\mathcal{H}) \text{ a.s.} \qquad \blacksquare$$

⚠ In general if $\mathcal{G}$ and $\mathcal{H}$ are not ordered, we do not have

$$E\big(E(X \mid \mathcal{G}) \mid \mathcal{H}\big) = E(X \mid \mathcal{G} \cap \mathcal{H}) \text{ a.s.}$$

(see exercises)

5 $\mathcal{G}$-MEASURABLE RVS. BEHAVE LIKE CONSTANTS.

$\mathcal{G} \subset \mathcal{F}$ fixed σ-algebra.

Prop. Let X, Y r.vs s.t. $X, Y \geq 0$ or $X, XY \in L^1$.
If Y is $\mathcal{G}$-measurable, then

$$E(XY \mid \mathcal{G}) = Y\, E(X \mid \mathcal{G}) \text{ a.s.}$$

Proof. Case $X, Y \geq 0$

- $Y\,E(X \mid \mathcal{G})$ is $\mathcal{G}$-measurable
- Let $Z \geq 0$ $\mathcal{G}$-measurable

Since $YZ \geq 0$ $\mathcal{G}$-measurable, we have

$$E(Z \cdot XY) = E(ZY \cdot X)$$

$$= E\big(ZY\,E(X \mid \mathcal{G})\big)$$

Hence $E(XY \mid \mathcal{G}) = Y\,E(X \mid \mathcal{G})$ a.s.

Case $X, XY \in L^1$

Let $X = X_+ - X_-$ $Y = Y_+ - Y_-$

$$E(XY \mid \mathcal{G}) = E(X_+Y_+ + X_-Y_- - X_+Y_- - \cancel{X}_-Y_+ \mid \mathcal{G})$$

$$\overset{\text{lin}}{=} (Y_+ - Y_-)E(X_+ \mid \mathcal{G}) - (Y_+ - Y_-)E(X_- \mid \mathcal{G})$$

$$= Y \; E(X \mid \mathcal{G})$$

∎

Application X, Y iid Ber$(\frac{1}{2})$ $E(XY \mid Y) = \frac{1}{2} Y$ a.s.

6 INDEPENDENT INFORMATION DOES NOT HELP.

or ≥ 0

Prop: Let $X \in L^1$. $\mathcal{G}, \mathcal{H} \subset \mathcal{F}$ σ-algebras.
If $\mathcal{H}$ is independent of $\sigma(\sigma(X) \cup \mathcal{G})$,
then
$$E(X \mid \sigma(\mathcal{G} \cup \mathcal{H})) = E(X \mid \mathcal{G}).$$

Rk: This generalizes the basic property $E(X \mid \mathcal{H}) = E(X)$ if X independent of $\mathcal{H}$.

Proof. $\{A \cap B, A \in \mathcal{G}, B \in \mathcal{H}\}$ is a π-system generating $\sigma(\mathcal{G} \cup \mathcal{H})$.

Let $A \in \mathcal{G}$, $B \in \mathcal{H}$.

We have
$$E(1_{A\cap B} X) = E(1_A 1_B X)$$
$$= E(1_A)\, E(1_B X)$$
$$= E(1_A)\, E(1_B E(X \mid \mathcal{G}))$$
$$= E(1_{A\cap B}\, E(X \mid \mathcal{G}))$$

Since $E(X \mid \mathcal{G})$ is $\sigma(\mathcal{G} \cup \mathcal{H})$-measurable, this concludes the proof.

7 MONOTONE CONVERGENCE

Thm. Let $X_n, n \geq 1$ be an increasing sequence of r.v. in $[0, \infty]$ with $X = \lim_{n\to\infty} \uparrow X_n$.
Let $\mathcal{G} \subset \mathcal{F}$ σ-algebra.

$$E(X \mid \mathcal{G}) = \lim_{n\to\infty} \uparrow E(X_n \mid \mathcal{G}) \quad \text{a.s.}$$

Proof. By monotonicity $(E(X_n \mid \mathcal{G}))_{n\geq 1}$ is an increasing sequence.
Let $X' = \lim_{n\to\infty} \uparrow E(X_n \mid \mathcal{G})$

- X' is $\mathcal{G}$-measurable (as a limit of $\mathcal{G}$-measurable random variables)

- Let $Z \geq 0$ $\mathcal{G}$-measurable.

For every $n \geq 1$, we have

$$E(Z X_n) = E(Z E(X_n | \mathcal{G}))$$

Hence, by monotone convergence,

$$E(Z X) = E(Z X') .$$

■

Thm (Conditional Fatou).

Let X_n, $n \geq 1$ be a sequence of r.v. in $[0, \infty]$

Let $\mathcal{G} \subset \mathcal{F}$ σ-algebra.

$$\liminf E(X_n | \mathcal{G}) \geq E(\liminf X_n | \mathcal{G}) \text{ a.s.}$$

Proof. For $i \geq n \geq 1$

$$E(\inf_{k \geq n} X_k | \mathcal{G}) \leq E(X_i | \mathcal{G}) \quad \text{a.s.}$$

Hence

$$E(\inf_{k \geq n} X_k | \mathcal{G}) \leq \inf_{k \geq n} E(X_k | \mathcal{G}) \text{ a.s.}$$

The result follows from (conditional) monotone cv. ■

8 DOMINATED CONVERGENCE.

Thm. Let $X, X_n, n \geq 1$ real random variables satisfying

- $X_n \xrightarrow[n\to\infty]{a.s} X$
- $\exists Z \geq 0$ in L^1 s.t $\forall n$ $|X_n| \leq Z$.

Then $E(X_n | \mathcal{G}) \longrightarrow E(X|\mathcal{G})$ a.s. and in L^1.

Proof. By applying Fatou's lemma to $Z + X_n \geq 0$, we get

$$E(Z + X | \mathcal{G}) = E(\liminf (Z + X_n) | \mathcal{G}) \quad \text{a.s.}$$

$$\leq E(Z|\mathcal{G}) + \liminf E(X_n | \mathcal{G}) \quad \text{a.s}$$

Thus $E(X|\mathcal{G}) \leq \liminf E(X_n | \mathcal{G})$ a.s.

Equivalently, by applying Fatou's lemma to $Z - X_n \geq 0$ we get $E(X|\mathcal{G}) \geq \limsup E(X_n|\mathcal{G})$ a.s.

The L^1-convergence follows from the domination

$$|E(X_n | \mathcal{G})| \leq E(Z | \mathcal{G}) \quad \text{a.s.}$$

(because $X_n \leq Z$ a.s and $-X_n \leq$ a.s).

which implies that $(E(X_n|\mathcal{G}))_{n\geq 1}$ is UI.

9 CONDITIONAL JENSEN.

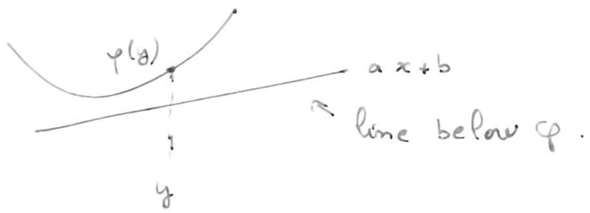
Thm: Let $\mathcal{G} \subset \mathcal{F}$ σ-algebra.

Let $X \in L^1$. Let $\varphi : \mathbb{R} \to \mathbb{R}_+$ convex

$$\varphi(E(X \mid \mathcal{G})) \leq E(\varphi(X) \mid \mathcal{G}) \quad \text{a.s.}$$

Proof. We use that for every $y \in \mathbb{R}$

$$\varphi(y) = \sup\{ay+b : a, b \in \mathbb{Q} \text{ s.t. } \forall x \ ax+b \leq \varphi(x)\}.$$



(exercise)

Let $a, b \in \mathbb{Q}$ s.t. $\forall x \in \mathbb{R}$ $ax+b \leq \varphi(x)$.

$$E(\varphi(X) \mid \mathcal{G}) \overset{\text{(linearity)}}{\geq} a\,E(X \mid \mathcal{G}) + b \quad \text{a.s.}$$

By taking the supremum over all $a, b \in \mathbb{Q}$ as above, we get the desired result.