CHAPTER 12

MARTINGALES

Goals: • formal definition of martingales :
• intuition/examples : "fair games" "loosing/winning games"
• basic properties
• transformation / construction of martingales.

Setup: • $(\Omega, \mathcal{F}, P)$ proba. space.
• $L^1 = \{ X : \Omega \to \mathbb{R}$ measurable $E(|X|) < \infty \}$.
•

1 FILTRATION.

Def. A filtration (on $(\Omega, \mathcal{F}, P)$) is an increasing sequence $(\mathcal{F}_n)_{n \geq 0}$ of sub σ-algebras of $\mathcal{F}$:

$$\mathcal{F}_0 \subset \mathcal{F}_1 \subset \dots \subset \mathcal{F}.$$

Terminology. $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n \geq 0}, P)$ is called a filtered proba space.

Intuition : $\mathcal{F}_n$ = "information at time n".

$\mathcal{F}_n$-measurable events can be observed at time n.

Notation: $\mathcal{F}_\infty = \sigma\left(\bigcup_{n\geq 0} \mathcal{F}_n\right)$. "information at the end"

Rk: $\bigcup_{n\geq 0} \mathcal{F}_n$ is a π-system generating $\mathcal{F}_\infty$.

Example 1 $\Omega = [0,1)$ $\mathcal{F} = \mathcal{B}([0,1))$

$$\mathcal{F}_n = \sigma\left(\left[\frac{k-1}{2^n}, \frac{k}{2^n}\right), k \in \{1, \dots, 2^n\}\right)$$

"dyadic filtration on $[0,1)$"

Example 2 : $(X_n)_{n\geq 0}$ random variables (in an arbitrary metric space)

$$\mathcal{F}_n = \sigma(X_0, X_1, \dots, X_n)$$

"canonical filtration of $(X_n)_{n\geq 0}$"

Terminology: • random process : sequence $(X_n)_{n\geq 0}$ of r.v.s.

• $(X_n)_{n\geq 0}$ is adapted (to $(\mathcal{F}_n)_n$) : $\forall n \geq 0$ X_n is $\mathcal{F}_n$-meas.

2 DEFINITION

Let $(X_n)_{n \geq 0}$ be a sequence of real r.v s.t. (for every $n \geq 0$) $X_n \in L^1$ and X_n is $\mathcal{F}_n$-measurable. It is called

- a $(\mathcal{F}_n)$-martingale if

$$\forall n \geq 0 \qquad E(X_{n+1} \mid \mathcal{F}_n) = X_n \quad \text{a.s}$$

- a $(\mathcal{F}_n)$-supermartingale if

$$\forall n \geq 0 \qquad E(X_{n+1} \mid \mathcal{F}_n) \leq X_n \quad \text{a.s.}$$

- a $(\mathcal{F}_n)$-submartingale if

$$\forall n \geq 0 \qquad E(X_{n+1} \mid \mathcal{F}_n) \geq X_n \quad \text{a.s.}$$

<u>Example 1</u> $X_n = x_n$ a.s. where $x_n \in \mathbb{R}$ deterministic

X_n martingale $\Leftrightarrow$ $\forall n \quad x_{n+1} = x_n$ "cst sequence"

X_n supermartingale $\Leftrightarrow$ $\forall n \quad x_{n+1} \leq x_n$ "$\searrow$ sequence"

X_n submartingale $\Leftrightarrow$ $\forall n \quad x_{n+1} \geq x_n$ "$\nearrow$ sequence"

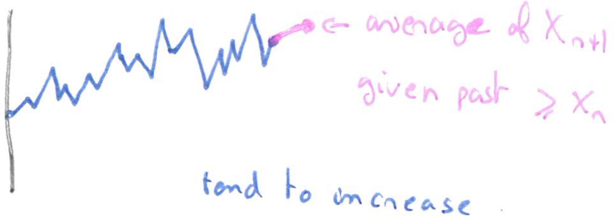
General intuition

Martingale = "fair game"

Supermartingale = "losing game"

↳ Durrett: "there is nothing super about a supermartingale"

Submartingale = "winning game"



"Supermartingale" ⟺ above its limit "super"

"Sub martingale" ⟺ below its limit "sub"

Rk on the Terminology: $f : \mathbb{R}^2 \to \mathbb{R}$

f harmonic ⟺ $f(x) = \frac{1}{|B(0,r)|} \int_{B(0,r)} f(y)\, dy$,

f superharmonic ⟺ $f(x) \geq \frac{1}{|B(0,r)|} \int_{B(0,r)} f(y)\, dy$,

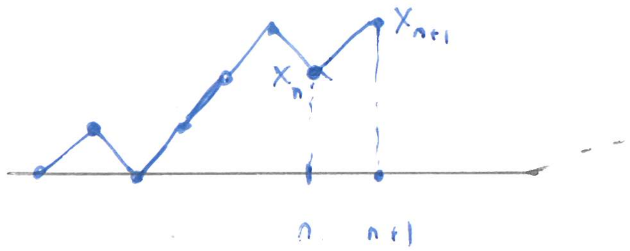
f subharmonic ⟺ $f(x) \leq \frac{1}{|B(0,r)|} \int_{B(0,r)} f(y)\, dy$.

Example 2 Simple RW.

$$(Z_n)_{n \geq 1} \text{ iid} \qquad P(Z_n = -1) = P(Z_n = +1) = \frac{1}{2}$$

$$X_n = Z_1 + Z_2 + \cdots + Z_n .$$

$$\mathcal{F}_n = \sigma(Z_1, \dots, Z_n)$$



$$\begin{aligned} E(X_{n+1} \mid \mathcal{F}_n) &= E(X_n + Z_{n+1} \mid \mathcal{F}_n) \\ &= E(X_n \mid \mathcal{F}_n) + E(Z_{n+1} \mid \mathcal{F}_n) \\ &= X_n + \underbrace{E(Z_{n+1})}_{=0} \end{aligned}$$

X_n is $\mathcal{F}_n$-meas.
Z_{n+1} is indep. of $\mathcal{F}_n$

Example 3: p-biased RW. $p \in [0,1]$

Same as above with $P(Z_n = +1) = p$ $\quad P(Z_n = -1) = 1-p$

$X_n = Z_1 + \dots + Z_n$ $\qquad$ super martingale if $p \leq \frac{1}{2}$

sub martingale if $p \geq \frac{1}{2}$

Rk1 • The definition of a martingale depends on the filtration $(\mathcal{F}_n)_{n\geq 0}$

Canonical filtration.

Let $(X_n)_{n\geq 0}$ be a $(\mathcal{F}_n)$-martingale.

$\mathcal{G}_n = \sigma(X_0, \dots, X_n)$. "canonical filtration"

$(X_n)_{n\geq 0}$ is a $(\mathcal{G}_n)$-martingale because

$$E(X_{n+1} \mid \mathcal{G}_n) \overset{\text{tower prop.}}{=} E\big(E(X_{n+1} \mid \mathcal{F}_n) \mid \mathcal{G}_n \big)$$

$$= E(X_n \mid \mathcal{G}_n)$$

$$= X_n \text{ a.s.}$$

"smallest filtration making (X_n) a martingale".

(same remark for supermart. / sub mart.)

Rk2 $\big((X_n)$ supermartingale$\big) \iff \big((-X_n)$ submartingale$\big)$

results for supermartingale $\longleftrightarrow$ results for sub martingale

Rk3 (X_n) martingale $\iff$ (X_n) supermartingale & sub martingale.

3 BASIC PROPERTIES.

Prop. Let $(\mathcal{F}_n)$ be a filtration.

If $(X_n)_{n \geq 0}$ is a $(\mathcal{F}_n)$-martingale, then

① $\forall n \geq k \quad E(X_n \mid \mathcal{F}_k) = X_k$ a.s.

② $\forall n \geq 0 \quad E(X_n) = E(X_0)$ "cst expectation"

If $(X_n)_{n \geq 0}$ is a $\mathcal{F}_n$-supermartingale, then

① $\forall n \geq k \quad E(X_n \mid \mathcal{F}_k) \leq X_k$ a.s.

② $\forall n \geq 0 \quad E(X_n) \leq E(X_0)$

Proof (martingale case)

① Let $k \geq 0$. The case $n = k$ follows from def.

Let $\ell \geq 0$.

$$E(X_{k+\ell+1} \mid \mathcal{F}_k) = E\big(E(X_{k+\ell+1} \mid \mathcal{F}_{k+\ell}) \mid \mathcal{F}_k\big)$$

$$= E(X_{k+\ell} \mid \mathcal{F}_k). \text{ a.s.}$$

Hence, the result follows by induction.

② For $n \geq 0$ $E(X_n) = E(E(X_n | \mathcal{F}_0))$
$$\stackrel{①}{=} E(X_0)$$

4 TRANSFORMATIONS

Prop. Let $\varphi : \mathbb{R} \to [0,\infty]$ be a convex function. Let $(\mathcal{F}_n)$ be a filtration. Let $(X_n)_{n\geq 0}$ r.v.s s.t for every $n \geq 0$ X_n is $\mathcal{F}_n$-measurable and $E(\varphi(X_n)) < \infty$.

(i) If $(X_n)_{n\geq 0}$ martingale, then $(\varphi(X_n))_{n\geq 0}$ is a submartingale

(ii) Assume in addition that φ is non decreasing. If $(X_n)_{n\geq 0}$ submartingale then $(\varphi(X_n))_{n\geq 0}$ submartingale.

"convexity makes you win"

Proof: (i) $\varphi(X_n) \in L^1$. By Jensen inequality.

$$\forall n \geq 0 \quad E(\varphi(X_{n+1}) | \mathcal{F}_n) \geq \varphi(E(X_{n+1} | \mathcal{F}_n))$$
$$= \varphi(X_n) \quad \text{a.s.}$$

(ii) $\forall n \geq 0 \quad E\left[\varphi(X_{n+1}) \mid \mathcal{F}_n\right] \geq \varphi\left(E(X_{n+1} \mid \mathcal{F}_n)\right)$

$$\underset{\underset{\varphi \nearrow}{\uparrow}}{\geq} \varphi(X_n) \quad \text{a.s.}$$

Applications. • Let $(X_n)_{n \geq 0}$ be a martingale, then

(X_n^+), $(|X_n|)$, $\underset{\underset{\text{if } E(X_n^2) < \infty}{\uparrow}}{(X_n^2)}$ are submartingale.

• Let $(X_n)_{n \geq 0}$ be a non-negative martingale, then $(\sqrt{X_n})_{n \geq 0}$ is a supermartingale.

($\sqrt{\cdot}$ is concave)

Ex: For the SRW $X_n = Z_1 + \dots + Z_n$ $\quad Z_i$ iid $\mathcal{U}(\{-1, 1\})$.

$E(X_n) = 0$ cst "X_n martingale"

$E(X_n^2) = n \nearrow$ "X_n^2 submartingale"

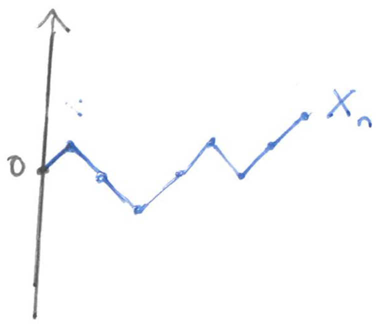
5 GAMBLING ON A COIN FLIP.

Let $(Z_n)_{n\geq 1}$ iid $P(Z_n=+1)=P(Z_n=-1)=\frac{1}{2}$.

We think of $Z_1, Z_2 \dots$ as n coin flips. At each step we bet 1 CHF. The money we have won after n steps is

$$X_n := Z_1 + \dots + Z_n .$$

($X_n < 0$ means that we lost money)



Fair game: For every $n \geq 0$ $E(X_n) = E(X_0) = 0$.

Can we change our betting strategy to win money?

An alternative strategy: We bet depending on the past:

- Start betting 1 CHF:

$$H_1 := 1$$

- As long as we are loosing we double our bet, and we stop as soon as we win.

For $n \geq 1$ $\quad H_n := \begin{cases} 2^{n-1} & \text{if } (Z_1,\dots,Z_n)=(-1,\dots,-1) \\ 0 & \text{otherwise} \end{cases}$

- Money won at step n: $Y_n = H_1 Z_1 + H_2 Z_2 + \dots + H_n Z_n$.

Suppose our first win is at step $k+1$ $(k \in \mathbb{N})$

$$z_1 = -1, \ldots, z_k = -1, z_{k+1} = +1.$$

Eventually we win:

$$\underbrace{-1-2-4\cdots-2^{k-1}}_{=-(2^k-1)} + 2^k = +1 \text{ CHF}.$$

$\Rightarrow$ We always win 1 CHF !

The strategy also works if $\begin{cases} P(z_n = +1) = \frac{1}{4} \\ P(z_n = -1) = \frac{3}{4}. \end{cases}$

So we can beat a casino ?

<u>Yes</u> : • if we have an infinite amount of money, and

• if we can play infinitely many times.

<u>No</u> • if we have to stop at a fixed time t

$\Rightarrow$ • $P(Y_t = 1-2^t) = P(z_1 = \ldots = z_t = -1)$

$$= \frac{1}{2^t}$$

• $P(Y_t = +1) = 1 - \frac{1}{2^t}$

$$E(Y_t) = (1-2^t) \times \frac{1}{2^t} + 1 \times \left(1 - \frac{1}{2^t}\right) = \underline{0}$$ "remains fair"

If we start with $2^\ell - 1$ CHF, we can play only ℓ times and the strategy does not work.

6. GAMBLING SYSTEM.

Question: Given a fair game (or a loosing game), can we bet to make it a winning game?

<u>Notation</u> Given $(H_n)_{n\geq 1}$, $(X_n)_{n\geq 0}$ r.v.s, we write

$$(H\cdot X)_n := \sum_{i=1}^{n} \underbrace{H_i}_{\text{modification of the bet}} \underbrace{(X_i - X_{i-1})}_{\text{money won at step } i}$$

<u>Thm:</u> Let $(\mathcal{F}_n)_{n\geq 0}$ filtration, $(H_n)_{n\geq 1}$ sequence of r.v.s s.t. H_n is $\mathcal{F}_{n-1}$-measurable and bounded* for all $n \geq 1$.

① If $(X_n)_{n\geq 0}$ is a $(\mathcal{F}_n)_{n\geq 0}$-martingale, then $H\cdot X$ is also a martingale

② If $H_n \geq 0$ for all $n\geq 1$ and (X_n) is a $(\mathcal{F}_n)$-super/sub martingale, then $H\cdot X$ is also a super/sub martingale.

<u>Interpretation</u>: However we bet, a martingale stays fair.

(* i.e. $\exists\, C_n$ cst s.t. $|H_n| \leq C_n$ a.s.)

Proof. ① • Since $H_1, \dots, H_n$ are bounded $(H\cdot X)_n \in L^1$.

• $(H \cdot X)_n$ is $\mathcal{F}_n$-measurable.

• Let $n \geq 0$

$$(H\cdot X)_{n+1} - (H\cdot X)_n = H_{n+1}\left(X_{n+1} - X_n\right) \text{ a.s.}$$

Hence

$$E\left((H\cdot X)_{n+1} \mid \mathcal{F}_n\right) - (H\cdot X)_n = H_{n+1} \underbrace{E\left(X_{n+1} - X_n \mid \mathcal{F}_n\right)}_{=0 \text{ a.s.}} \text{a.s} \quad \blacksquare$$

② Exercise (similar proof).

Rk: in particular $E\left((H\cdot X)_n\right) = E(X_0)$ for every n (martingale case).

7. STOPPING TIMES.

Motivation: Given a fair-game, can we stop a well-chosen time to our advantage?

Def. A random variable T with values in $\mathbb{N} \cup \{+\infty\}$ is called a stopping time (of the filtration $(\mathcal{F}_n)_{n\geq 0}$) if

$$\boxed{\forall n \in \mathbb{N} \quad \{T = n\} \in \mathcal{F}_n.}$$

Interpretation:

- T = "time at which we want to stop the process"
- $\{T=n\} \in \mathcal{F}_n$ "looking only at the information available at time n, we can decide if we stop (i.e. $T=n$) or not ($T \neq n$)"

Prop : Let T be a r.v. with values in $\mathbb{N} \cup \{+\infty\}$.

$$\left(T \text{ is a } (\mathcal{F}_n)\text{-stopping time}\right) \Leftrightarrow \left(\forall n \geq 0 \quad \{T \leq n\} \in \mathcal{F}_n\right)$$

Proof. $\boxed{\Rightarrow}$ $\{T \leq n\} = \bigcup_{k \leq n} \underbrace{\{T=k\}}_{\in \mathcal{F}_n}$

$\boxed{\Leftarrow}$ $\{T=n\} = \underbrace{\{T \leq n\}}_{\in \mathcal{F}_n} \cap \underbrace{\{T \leq n-1\}}_{\in \mathcal{F}_n}{}^c$.

Prop. Let T be a $(\mathcal{F}_n)$-stopping time.

$$\{T=+\infty\} \in \mathcal{F}_\infty$$

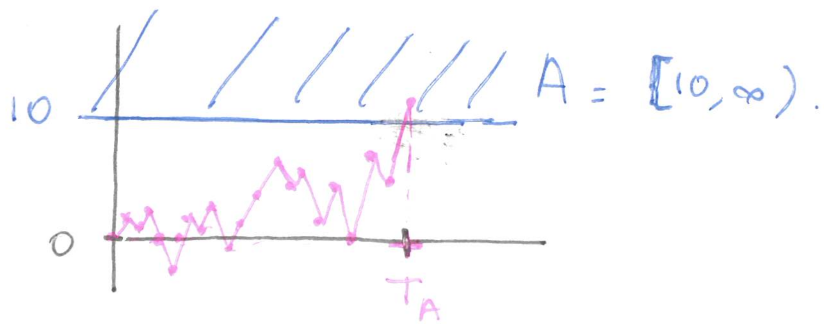
Proof. $\{T=+\infty\} = \bigcap_{n \in \mathbb{N}} \{T=n\}^c$ ■

Examples. • $T=10$ (cst) is a stopping time $\left(\{T=n\} = \begin{cases} \emptyset & \text{if } n\neq 10 \\ \Omega & \text{if } n=10 \end{cases} \right)$

• **Hitting time:** Let $(X_n)_{n\geq 0}$ be real r.v.s s.t. X_n is $\mathcal{F}_n$-measurable, $A \in \mathcal{B}(\mathbb{R})$.

$$T_A := \min \{ n : X_n \in A \}$$

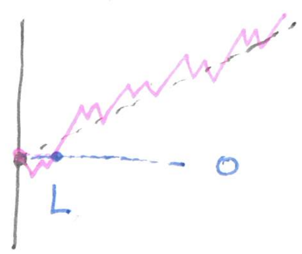
is a stopping time. (convention $\min \emptyset = +\infty$)



Indeed: for every n, we have

$$\{T_A = n\} = \{X_0 \notin A, \dots, X_{n-1} \notin A, X_n \in A\} \in \mathcal{F}_n$$

Non-example: $Z_1, Z_2, \dots$ iid $P(Z_i = +1) = \frac{3}{4}$ $P(Z_i = -1) = \frac{1}{4}$.



L = last visit time of 0 of $S_n = Z_1 + \dots + Z_n$.

$\mathcal{F}_n = \sigma(Z_1, \dots, Z_n)$

(exercise)

↳ idea $\{L = n\}$ depends on $Z_{n+1}, Z_{n+2}, \dots$.

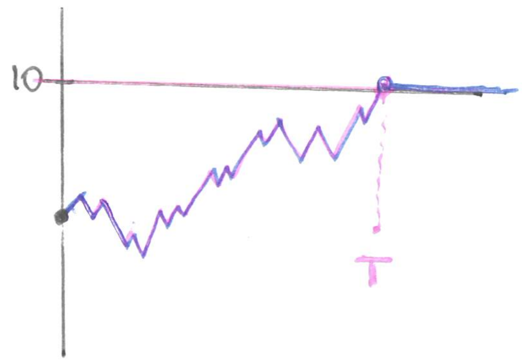
8 STOPPED MARTINGALE.

Example: $X_n = Z_1 + \dots + Z_n$ where (Z_i) iid $P(Z_i=+1)=P(Z_i=-1)=\frac{1}{2}$.

We play until we gain 10 CHF.

$$T := \min\{n \geq 0 : X_n \geq 10\}$$

(Hitting time of $A = [10, \infty)$)



Since $\limsup X_n = +\infty$ a.s., we know $T < \infty$ a.s.

↳ so we will gain 10 CHF eventually.

Did we manage to create a winning strategy?

NB: the money we gained at time n with this strategy is

$$X_{n \wedge T} = \begin{cases} X_n & \text{if } n < T \\ 10 & \text{if } n \geq T \end{cases}$$

Prop. Let $(X_n)_{n\geq 0}$ be a (sub/super/-) martingale. Let T be a stopping time. Then

$(X_{n\wedge T})_{n\geq 0}$ is a (sub/super/-) martingale.

Proof. For every $n \geq 1$ set

$$H_n = 1_{\{n \leq T\}}$$

"we bet only if $n \leq T$"

Since $H_n = 1 - 1_{\{T < n\}} = 1 - 1_{\{T \leq n-1\}}$, we have that H_n is $\mathcal{F}_{n-1}$-measurable.

$$X_{n\wedge T} = X_0 + \sum_{i=1}^{n\wedge T} X_i - X_{i-1}$$

$$= X_0 + \sum_{i=1}^{n} H_i (X_i - X_{i-1})$$

$$= X_0 + (H \bullet X)_n$$

Since X_0 is $\mathcal{F}_0$-measurable, the constant sequence $Y_n = X_0$ is a $(\mathcal{F}_n)$-martingale. The proof follows because the sum of two martingales (resp. submart., supermart.) is a martingale (resp. submart., supermart.).

Back to the question : "stopping at 10 CHF" is a winning strategy?

Yes → if we have infinite amount of money and we can play infinitely many times.

No → if we can play only finitely many times.

$$\forall n \geq 0 \quad E(X_{n \wedge T}) = 0$$

9 CLOSED MARTINGALES.

Prop. Let $X \in L^1$.

$X_n := E(X \mid \mathcal{F}_n)$ is a $\mathcal{F}_n$-martingale.

(called "closed martingale")

Proof: • $X_n \in L^1$, X_n is $\mathcal{F}_n$-measurable for every $n \geq 0$

• For $n \geq 0$, by the tower property, we have.

$$E(X_{n+1} \mid \mathcal{F}_n) = E(E(X \mid \mathcal{F}_{n+1}) \mid \mathcal{F}_n)$$
$$= E(X \mid \mathcal{F}_n) = X_n \text{ a.s.} \quad \blacksquare$$