CHAPTER 13

ALMOST SURE CONVERGENCE.

Goal: Durrett "one cannot make money with a martingale but one can prove theorems"

Setup: . $(\Omega, \mathcal{F}, P)$ proba space.

. $(\mathcal{F}_n)_{n\geq 0}$ filtration

I MAIN THEOREM.

Thm: Let $(X_n)_{n\geq 0}$ be a submartingale, a supermartingale, or a martingale. If

$$\sup_{n\geq 0} E(|X_n|) < \infty, \quad \text{"}(X_n) \text{ bounded in } L^1\text{"}$$

then $(X_n)_{n\geq 0}$ converges almost surely to a limit $X_\infty \in L^1$.

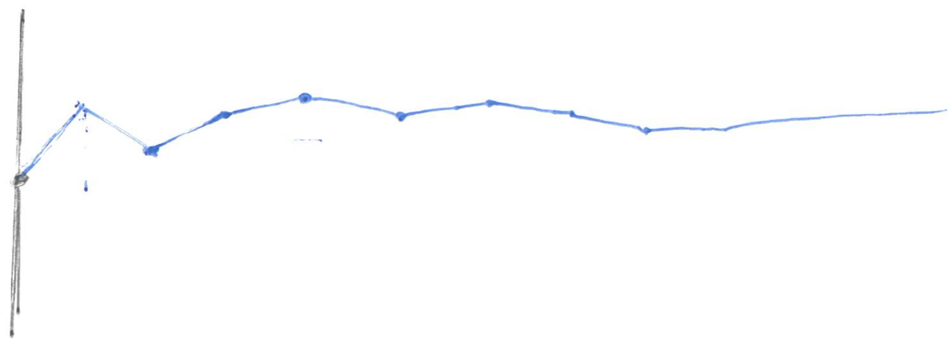
Rk: generalisation of $(u_n)_n$ monotone bounded $\Rightarrow$ $(u_n)_n$ converges.

<u>Example</u> : $Z_1, Z_2, \dots$ iid $\mathcal{U}(\{+1,-1\})$. $\mathcal{F}_n = \sigma(Z_1, \dots, Z_n)$

- $X_n = \sum_{k=1}^{n} \frac{1}{k} Z_k$

$$E(|X_n|) \leq \sqrt{E(X_n^2)} = \sqrt{\sum_{k=1}^{n} \frac{1}{k^2}} \leq \frac{\pi}{\sqrt{6}}$$

(X_n) c.v. a.s.



- $X_n = \sum_{k=1}^{n} Z_k$ does not cv a.s. $\left(E(|X_n|) \simeq \sqrt{n} \right)$

<u>Corollary</u>

- If (X_n) is a submartingale (or a martingale) and $\sup_{n \geq 0} E(X_n^+) < \infty$, then (X_n) cv as to a limit $X_\infty \in L^1$

"generalization of $(u_n) \nearrow$ upper bounded $\Rightarrow (u_n)$ cv."

- If (X_n) is a supermartingale (or a martingale) and $\sup_{n \geq 0} E(X_n^-) < \infty$, then (X_n) cv a.s to a limit $X_\infty \in L^1$

"generalization of $(u_n) \searrow$ lowerbounded $\Rightarrow (u_n)$ cv"

Pf (of Corollary). submartingale case. For every $n \geq 1$

$$E(X_0) \leq E(X_n) = E(X_n^+) - E(X_n^-).$$

Hence $E(X_n^+) \leq E(|X_n|) \leq 2E(X_n^+) - E(X_0)$.

Therefore $\sup_{n\geq 0} E(X_n^+) < \infty \iff \sup_{n \geq 0} E(|X_n|) < \infty$ ∎

" if $(u_n)\uparrow$, (u_n upper bounded) $\iff$ ($|u_n|$ bounded) "

Rk1. In particular a non negative martingale or supermartingale converges almost surely.

Rk2: If a sequence $u_1, u_2, \dots \in \mathbb{Z}$ converges then it is stationary. Hence if a martingale takes values in $\mathbb{Z}$ and is bounded in L^1 then it must be stationary almost surely; ie

$$P(\exists n_0 \; \forall n \geq n_0 \; X_n = X_{n_0}) = 1.$$

2 AN APPLICATION

Let $Z_1, Z_2, \dots$ iid $P(Z_i=+1)=P(Z_i=-1)=\frac{1}{2}$.

$S_n = Z_1+\dots+Z_n$.

$$\limsup S_n = +\infty \quad \text{a.s.}$$

(alternative proof)

Pf. Let $k\geq 1$, $T=\min\{n : S_n = k\}$ stopping time.

$(S_{n\wedge T})_{n\geq 0}$ is a martingale in $\mathbb{Z}$ and

$$E(S_{n\wedge T}^{+}) \leq k$$

Hence $(S_{n\wedge T})_{n\geq 1}$ is a.s. stationary. This implies that $T<\infty$ a.s. (otherwise $S_{n\wedge T}=S_n$ is not stationary a.s). Therefore, for every $k\geq 1$

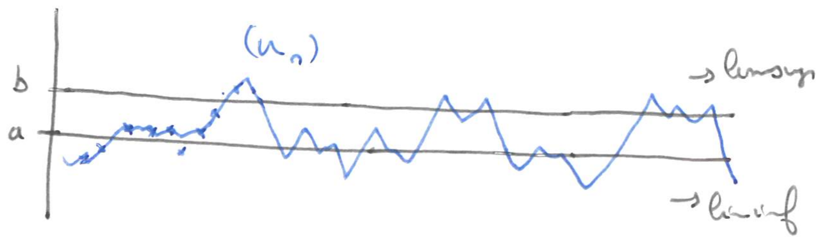
$$\mathbb{P}\left(\exists\, n\geq 1 \text{ s.t. } X_n = k\right) = 1 \quad \blacksquare$$

3 DETERMINISTIC CRITERION FOR CV. (Remiolic)

Let $(u_n)_{n\geq 0}$ be a sequence of real numbers.

$$\left(\underset{\text{in } [-\infty,+\infty]}{(u_n) \text{ does not cv}}\right) \Rightarrow \left(\exists a,b \in \mathbb{Q} \quad \liminf u_n < a < b < \limsup u_n\right)$$

$\Rightarrow$ ($\exists a < b$ $a, b \in \mathbb{Q}$ s.t. $(u_n)_{n\geq 0}$ crosses the intervall $[a,b]$ infinitely many times.)



In order to prove $(u_n)_{n\geq 0}$ converges we can do the following.

- define $N_{a,b}$ the number of crossings of $[a,b]$
- prove that $N_{a,b} < \infty$ for every $a < b$ $a, b \in \mathbb{Q}$.

4 DOOB UP-CROSSING INEQUALITY.

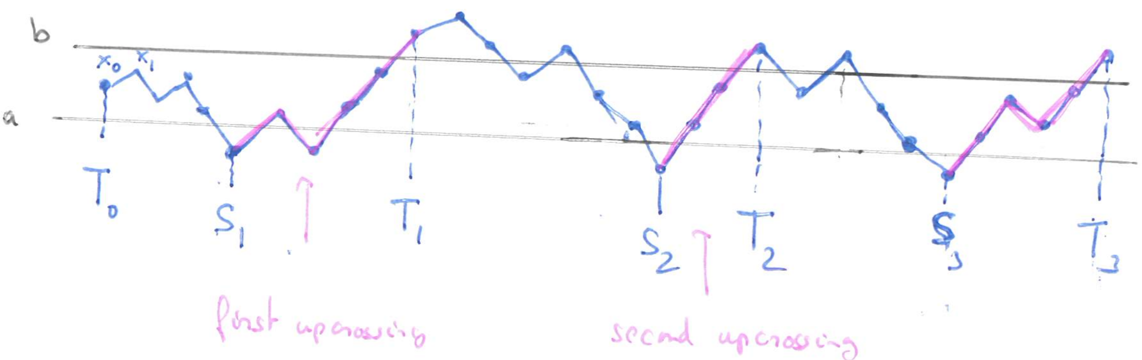
Let $(X_n)_{n\geq 0}$ be a sequence of real r.v.s s.t. X_n $\mathcal{F}_n$-meas., $a, b \in \mathbb{Q}$ $a < b$.

We set $T_0 = 0$ and define by induction (for $i \geq 1$)

$$S_i = \min\{ n \geq T_{i-1} \quad X_n \leq a \},$$

$$T_i = \min\{ n \geq S_i \quad X_n \geq b \}.$$

$(\min \emptyset = +\infty)$.



Number of upcrossings before time n.

$$N_n(a,b) := \sum_{i=1}^{\infty} 1_{T_i \leq n}.$$

Total number of upcrossings:

$$N(a,b) := \sum_{i=1}^{\infty} 1_{T_i < \infty}. \qquad (N = \lim_{n\to\infty} \uparrow N_n \text{ a.s})$$

Rk: For every i, S_i, T_i are stopping times. Indeed, for all n

$$\{T_i \leq n\} = \bigcup_{0 < k_1 < l_1 < \dots < k_i < l_i} \{X_{k_1} \leq a, X_{l_1} \geq b \dots\}$$

and a similar argument works for S_i.

Thm (Doob Upcrossing inequality).

Assume that (X_n) is a supermartingale.
For every $a < b$ and $n \in \mathbb{N}$, we have

$$(b-a)\, E\big(N_n(a,b)\big) \leq E\big((a - X_n)^+\big)$$

<u>Proof.</u> Define for $n \geq 1$

$$H_n = \sum_{i=1}^{\infty} 1_{S_i < n \leq T_i} \qquad (\leq 1)$$

"We bet 1 CHF during the upcrossings"

We have H_n $\mathcal{F}_{n-1}$-measurable. Indeed.

$$\forall i \geq 1 \quad \{S_i < n \leq T_i\} = \{S_i \leq n-1\} \cap \{T_i \leq n-1\}^c \in \mathcal{F}_{n-1}$$

Write $N_n = N_n(a,b)$.

$$\begin{aligned}
(H \cdot X)_n &= \sum_{k=1}^{n} H_k (X_k - X_{k-1}) \\
&= \sum_{k=1}^{n} \sum_{i=1}^{\infty} 1_{S_i < k \leq T_i} (X_k - X_{k-1}) \\
&= \sum_{i=1}^{\infty} \underbrace{\sum_{k=1}^{n} 1_{S_i < k \leq T_i} (X_k - X_{k-1})}_{\begin{cases} X_{T_i} - X_{S_i} & \text{if } T_i \leq n \quad (\text{i.e } i \leq N_n) \\ X_n - X_{S_i} & \text{if } S_i \leq n < T_i \quad (\text{i.e } i = N_n + 1) \\ 0 & \text{if } \ T_i < n \quad (\ldots) \end{cases}}.
\end{aligned}$$

Hence

$$(H\cdot X)_n = \underbrace{\sum_{i=1}^{N_n} X_{T_i} - X_{S_i}}_{\text{what we won so far}} + \underbrace{1_{S_{N_n+1}\leq n}\Big(X_n - \overbrace{X_{S_{N_n+1}}}^{\leq a}\Big)}_{\text{what we are currently betting in the unfinished up-crossing}}$$

$\geq N_n \cdot (b-a)$

$$\geq 1_{S_{N_n+1}\leq n}(X_n - a)$$

$$\geq 1_{S_{N_n+1}\leq n}\left[-(X_n-a)^-\right]$$

$$\geq -(X_n-a)^-$$

At time n
- we won at least $N_n(b-a)$
- we lost at most $(X_n-a)^-$

Since $H\cdot X$ is a super martingale, we have for every n $\quad E((H\cdot X)_n) \leq E((H\cdot X)_0) = 0$

Hence $(b-a)E(N_n) \leq E((X_n-a)^-)$ ∎

5 PROOF OF THE THEOREM.

Without loss of generality, we assume that $(X_n)_{n\geq 0}$ is a supermartingale s.t

$$C := \sup_{n\geq 0} E(|X_n|) < \infty .$$

Let $a, b \in \mathbb{Q}$ $a < b$. By Doob's upcrossing inequality.

$$E(N_n(a,b)) \leq \frac{1}{b-a} E((X_n - a)^-)$$

$$\leq \frac{1}{b-a} (C + |a|)$$

Since $N_n(a,b) \uparrow N_\infty(a,b)$ a.s, by monotone convergence, we have

$$E(N_\infty(a,b)) \leq \frac{1}{b-a}(C+|a|).$$

Therefore $N_\infty(a,b) < \infty$ a.s.

Assume for contradiction

$$P(\liminf X_n < \limsup X_n) > 0$$

Since $\{\liminf X_n < \limsup X_n\} \subset \bigcup_{\substack{a<b \\ a,b\in\mathbb{Q}}} \{N_\infty(a,b) = +\infty\}$

There must exist $a, b \in \mathbb{Q}$ s.t $P(N_\infty(a,b) = +\infty) > 0$ contradiction.

Hence $\liminf X_n = \limsup X_n$ a.s. Let X_∞ be the a.s. limit of (X_n) in $[-\infty, \infty]$. By Fatou's lemma

$$\begin{aligned} E(|X_\infty|) &= E(\liminf |X_n|) \\ &\leq \liminf E(|X_n| \\ &\leq C \end{aligned}$$

Hence $|X_\infty| < \infty$ a.s and $X_\infty \in L^1$.