CHAPTER 14.

UNIFORMLY INTEGRABLE MARTINGALES -

Goals: • Link cv in L^1 $\Leftrightarrow$ closed martingales.

• Optional stopping.

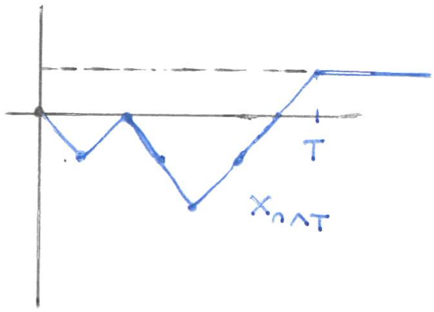
Setup. • $(\Omega, \mathcal{F}, P)$ probab. space

• $(\mathcal{F}_n)_{n \geq 0}$ filtration. $\mathcal{F}_\infty = \sigma\left(\bigcup_{n\geq 0} \mathcal{F}_n\right)$

INTRO / MOTIVATION

• $Z_1, Z_2, \dots$ iid $\mathcal{U}(\{+1,-1\})$. $X_n = Z_1 + \dots + Z_n$, $n \geq 1$

$T = \min\{n : X_n = +1\}$.



• Chap. 12: $(X_{n\wedge T})_{n\geq 0}$ martingale

$E(X_{n\wedge T}) = E(X_0) = 0$.

• Chap. 13 $X_{n\wedge T} \leq 1$

$X_{n\wedge T} \xrightarrow{a.s} X_T = +1$

↳ The cv does not hold in L^1 (otherwise $E(X_{n\wedge T}) \to E(X_T)$)

↳ $E(X_T) \neq E(X_0)$ → the martingale property does not generalize to random time without conditions.

• Chap 14: For a martingale: when $X_n \xrightarrow{L^1} X_\infty$?

For a stopping time: when $E(X_0) = E(X_T)$?

1 CONDITIONAL EXPECTATION AND UI

Prop. Let $X \in L^1$. Let $(\mathcal{G}_i)_{i \in I}$ be a collection of σ-algebras ($\mathcal{G}_i \subset \mathcal{F}$). The family $\{E(X \mid \mathcal{G}_i), i \in I\}$ is UI.

Proof. Write $X_i = E(X \mid \mathcal{G}_i)$. Let $a > 0$.

For every $i \in I$, we have

$$E(|X_i| 1_{|X_i| \geq a}) \leq E(E(|X| \mid \mathcal{G}_i) 1_{\underbrace{|X_i| \geq a}_{\in \mathcal{G}_i}})$$

def of conditional exp.
$$= E(|X| 1_{|X_i| \geq a})$$

$$= E(|X| 1_{|X| \geq \sqrt{a}, |X_i| \geq a}) + E(|X| 1_{|X| < \sqrt{a}, |X_i| \geq a})$$

$$\leq E(|X| 1_{|X| \geq \sqrt{a}}) + \sqrt{a}\, \underbrace{P(|X_i| \geq a)}_{\leq \frac{1}{a} E(|X_i|) \leq \frac{1}{a} E(|X|)}$$

Hence
$$\sup_{i \in I} E(|X_i| 1_{|X_i| \geq a}) \leq E(|X| 1_{|X| \geq a}) + \frac{1}{\sqrt{a}} E(|X|)$$

$$\xrightarrow[a \to \infty]{} 0$$

2 UI-MARTINGALES

Thm. Let $(X_n)_{n\geq 0}$ be a martingale. The following are equivalent:

(i) (X_n) converges a.s and in L^1 to a r.v. $X_\infty \in L^1$.

(ii) $\exists X \in L^1$ s.t. $\forall n \geq 0$ $X_n = E(X \mid \mathcal{F}_n)$. a.s

(iii) $(X_n)_{n\geq 0}$ is U.I.

When these conditions hold, one may take $X = X_\infty$ in (ii)

Proof. (i) $\Rightarrow$ (ii) Fix $n \geq 0$

For every $p \geq n$, we have $E(X_p \mid \mathcal{F}_n) = X_n$ a.s.

$\phi : X \mapsto E(X \mid \mathcal{F}_n)$ is a 1-Lip mapping from L^1 to L^1,

hence $X_p \xrightarrow{L^1} X_\infty \Rightarrow \phi(X_p) \xrightarrow{L^1} \phi(X_\infty)$

For $p \geq n$, we have

$$E\big(\,|E(X_\infty \mid \mathcal{F}_n) - X_n|\,\big) = E\big(\,|E(X_\infty \mid \mathcal{F}_n) - E(X_p \mid \mathcal{F}_n)|\,\big)$$

$$\leq E\big(E(|X_\infty - X_p| \mid \mathcal{F}_n)\big)$$

$$= E\big(|X_\infty - X_p|\big) \xrightarrow[p\to\infty]{} 0.$$

Hence $X_n = E(X_\infty \mid \mathcal{F}_n)$ a.s.

(ii) $\Rightarrow$ (iii) $\quad (E(X \mid \mathcal{F}_n))_{n \geq 0}$ is UI.

(iii) $\Rightarrow$ (i) $\quad$ If $(X_n)_{n\geq 0}$ is UI, then it is bounded in L^1.

Therefore, $X_n \xrightarrow{a.s} X_\infty$ where $X_\infty \in L^1$.

Since $(X_n)_{n \geq 0}$ is UI, it also implies $X_n \xrightarrow{L^1} X_\infty$.

(because $X_n \xrightarrow{a.s} X_\infty \Rightarrow X_n \xrightarrow{P} X_\infty$).

Rk: no uniqueness in (ii). ex: $X_n = 2$ cte $\mathcal{F}_n = \{\emptyset, \Omega\}$ any X with $E(X) = 2$ works.

3 CONVERGENCE OF CLOSED MARTINGALE.

Corollary. Let $X \in L^1$.

$$E(X \mid \mathcal{F}_n) \xrightarrow{a.s, L^1} E(X \mid \mathcal{F}_\infty)$$

Proof. By the theorem above, $(E(X \mid \mathcal{F}_n))$ cv a.s. and in L^1 to a r.v. $X_\infty \in L^1$. We need to show

$$X_\infty = E(X \mid \mathcal{F}_\infty).$$

It suffices to prove

$$\forall A \in \bigcup_{n \geq 0} \mathcal{F}_n \quad E(X_\infty 1_A) = E(X 1_A).$$

Let $A \in \mathcal{F}_n$ for some fixed $n \geq 0$. For every $p \geq n$, $A \in \mathcal{F}_p$, hence

$$E(X 1_A) = E\left(E(X \mid \mathcal{F}_p) 1_A\right) = \underbrace{E(X_p 1_A)}_{\substack{\downarrow p \to \infty \\ E(X_\infty 1_A)}}$$

(because $X_p \xrightarrow{L^1} X_\infty$).

■

Interest of the previous corollary: approximation of r.v. defined on a large σ-algebra.

Application: alternative proof of Kolmogorov 0-1 law.

$$\mathcal{T} = \bigcap_{n \geq 1} \sigma(Z_n, Z_{n+1}, \dots) \quad \text{where } Z_1, Z_2, \dots \text{ indep.}$$

$$\mathcal{F}_n = \sigma(Z_1, \dots, Z_n) \qquad (\mathcal{F}_0 = \{\emptyset, \Omega\}).$$

Let $A \in \mathcal{T}$. Since $A \in \mathcal{F}_\infty$, we have $E(1_A \mid \mathcal{F}_\infty) = 1_A$ a.s.

Hence

$$E(1_A \mid \mathcal{F}_n) \xrightarrow[n \to \infty]{a.s., L^1} 1_A \quad \text{a.s.} \quad \text{(Levy's 0-1 law).}$$

$$P(A) = E(1_A \cdot 1_A)$$

$$= \lim_{n \to \infty} E\Big(\underbrace{E(1_A \mid \mathcal{F}_n)}_{\in \sigma(Z_1, \dots, Z_n)} \cdot \underbrace{1_A}_{\in \sigma(Z_{n+1}, \dots)} \Big)$$

$$\overset{\text{indep.}}{=} \lim_{n \to \infty} E(E(1_A \mid \mathcal{F}_n)) \, P(A)$$

$$= P(A)^2.$$

4 STOPPED σ-ALGEBRA

<u>Def.</u> Let T be a stopping time. We set

$$\mathcal{F}_T = \{ A \in \mathcal{F}_\infty : \forall n \in \mathbb{N} \quad A \cap \{T=n\} \in \mathcal{F}_n \}.$$

<u>Rk:</u> $\mathcal{F}_T$ is a σ-algebra:

- $\Omega \in \mathcal{F}_T$ because T is a stopping time.
- $A_1, A_2, \dots \in \mathcal{F}_T \Rightarrow \forall n \geq 0 \quad \bigcup_i A_i \cap \{T=n\} \in \mathcal{F}_n$
- $A \in \mathcal{F}_T \Rightarrow \forall n \geq 0 \quad \{T=n\} \setminus (A \cap \{T=n\}) \in \mathcal{F}_n$

<u>Intuition</u> $\mathcal{F}_T$ = "information before time T"

$$\overset{\text{(Reasonable)}}{=} \sigma(\mathcal{F}_1 \cup \dots \cup \mathcal{F}_T)$$

<u>Example:</u> T is $\mathcal{F}_T$-measurable.

<u>Notation.</u> $(X_n)_{n \in \mathbb{N} \cup \{+\infty\}}$ r.v.s. $\qquad T$ r.v. in $\mathbb{N} \cup \{+\infty\}$

$$X_T = \sum_{n \in \mathbb{N} \cup \{+\infty\}} X_n \mathbb{1}_{T=n}$$

<u>Prop.</u> If X_n $\mathcal{F}_n$-meas for every $n \in \mathbb{N} \cup \{\infty\}$,

T stopping time,

then X_T is $\mathcal{F}_T$-measurable.

Proof. For every n $\quad X_n 1_{T=n}$ is $\mathcal{F}_\infty$-meas.

Hence X_T is $\mathcal{F}_\infty$-measurable.

Let $B \in \mathcal{B}(\mathbb{R})$.

- $\{X_T \in B\} \in \mathcal{F}_\infty$.

- For every $n \in \mathbb{N}$ $\{X_T \in B\} \cap \{T=n\} = \{X_n \in B\} \cap \{T=n\} \in \mathcal{F}_n$.

Hence $\{X_T \in B\} \in \mathcal{F}_T$ ∎

Prop. Let S, T be two stopping times.

$$(S \leq T) \Rightarrow (\mathcal{F}_S \subset \mathcal{F}_T).$$

Proof. Let $A \in \mathcal{F}_\infty$.

$$A \in \mathcal{F}_S \Leftrightarrow \forall n \in \mathbb{N} \quad A \cap \{S \leq n\} \in \mathcal{F}_n$$

$$\Rightarrow \forall n \in \mathbb{N} \quad A \cap \underbrace{\{S \leq n\} \cap \{T=n\}}_{= \{T=n\}} \in \mathcal{F}_n$$

$$\Leftrightarrow \forall n \in \mathbb{N} \quad A \cap \{T=n\} \in \mathcal{F}_n$$

$$\Leftrightarrow A \in \mathcal{F}_T$$ ∎

NEXT: OPTIONAL STOPPING THEOREMS.

Idea: $(X_n)_{n\geq 0}$ martingale, T stopping time.

Under which conditions on $(X_n)_{n\geq 0}$ and T do we have

$$E(X_T) = E(X_0)\ ?$$

We will see three results: this holds

1. T bounded, (X_n) arbitrary,
2. $T < \infty$ a.s, $(X_{n\wedge T})$ UI,
3. T arbitrary, (X_n) UI.

5 BOUNDED STOPPING TIME. (OPT. STOP. I)

Prop. Let $(X_n)_{n\geq 0}$ be a martingale.
Let T be a stopping time such that $T \leq k$ a.s for some $k \in \mathbb{N}$.
Then
$$E(X_T) = E(X_0).$$

Proof. Note that
$$X_{T\wedge k} = \sum_n X_{n\wedge k} 1_{T\wedge k = n}$$
$$= \sum_{n=0}^{k} X_n 1_{T=n}$$
$$= X_T \text{ a.s.}$$
Since $(X_{T\wedge n})$ martingale, we have $E(X_0) = E(X_{T\wedge k}) = E(X_T)$ ∎

To remember:

(X_n) arbitrary, T arbitrary: "optimal stopping" always works for the stopping time $T \wedge n$, n fixed.

Application

$Z_1, Z_2, \dots$ iid $\mathcal{U}(\{+1,-1\})$. $\mathcal{F}_n = \sigma(Z_1, \dots, Z_n)$.

$S_n = Z_1 + \dots + Z_n$ simple random walk on $\mathbb{Z}$.

quadratic martingale:

$$X_n = S_n^2 - n$$

$$E(X_{n+1} - X_n \mid \mathcal{F}_n) = E\left((S_n + Z_{n+1})^2 - S_n^2 - 1 \mid \mathcal{F}_n\right)$$

$$= 2E(S_n Z_{n+1} \mid \mathcal{F}_n)$$

$$= 2 S_n E(Z_{n+1} \mid \mathcal{F}_n) = 0 \text{ a.s.}$$

$$T = \min\{n : |S_n| = a\} \qquad (a \in \mathbb{N}).$$

For every $n \geq 0$

$$0 = E(X_{T\wedge n}) = E(S_{T\wedge n}^2) - E(T \wedge n)$$

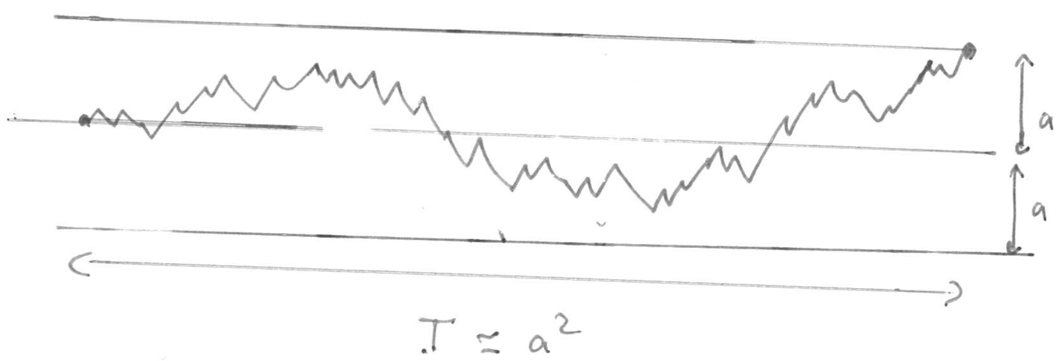
Since $T < \infty$ a.s., we have $S_{T\wedge n}^2 \xrightarrow{a.s} S_T^2 = a^2$

Furthermore, by dominated cv $(|S_{T\wedge n}^2| \leq a^2)$

we get $E(S_{T\wedge n}^2) \longrightarrow a^2$.

Since $E(T\wedge n) \underset{n\to\infty}{\longrightarrow} E(T)$ (monotone cv)

we get $E(T) = a^2$.



6 UI STOPPED MARTINGALE (OPT. STOP. II)

Prop. Let $(X_n)_{n\geq 0}$ be a martingale, T a stopping time such that $T < \infty$ a.s. and $(X_{n\wedge T})$ UI.

We have $\boxed{E(X_0) = E(X_T)}$

Proof. Since $T<\infty$ a.s., we have

$$X_{n\wedge T} \xrightarrow{a.s} X_T.$$

Since $(X_{n\wedge T})$ U.I the convergence also holds in L^1. Therefore $E(X_T) = \lim_{n\to\infty} \underbrace{E(X_{n\wedge T})}_{=E(X_0)}$ ∎

Rk: The conditions of the prop. hold: if

① $E(T) < \infty$ and $\exists C$ s.t. $\forall n$ $|X_{n+1} - X_n| \leq C$ a.s.

or

② $T < \infty$ a.s. and $\exists C$ s.t $\forall n$ $|X_{n \wedge T}| \leq C$ a.s.

Indeed ① implies the domination:

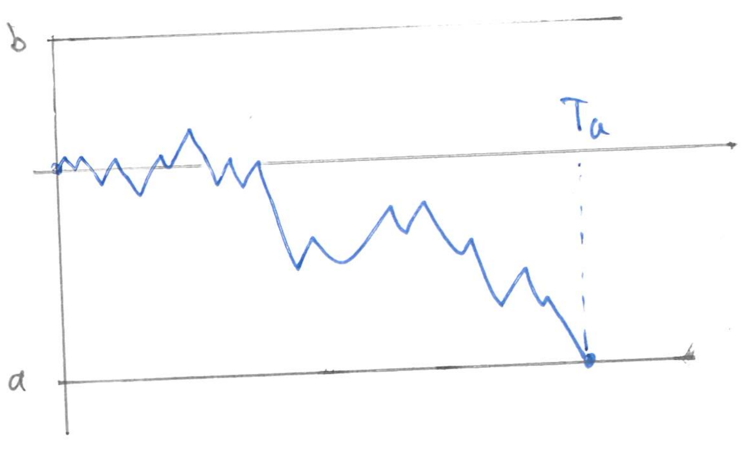
$$|X_{n \wedge T}| \leq C\, T$$

② gives the domination $|X_{n \wedge T}| \leq C$.

Application: $(S_n)_{n \geq 0}$ SRW on $\mathbb{Z}$.

$T_k = \min\{n : S_n = k\}$, $a < 0 < b$

$$P(T_a < T_b) = \frac{b}{b-a}$$ (exercise)



7 UI MARTINGALE (OPT. STOP. III).

<u>Thm</u> Let $(X_n)_{n\geq 0}$ be a UI martingale, $X_\infty = \lim X_n$ a.s.

For every stopping time $T : \Omega \to \mathbb{N} \cup \{+\infty\}$.

$$X_T = E(X_\infty \mid \mathcal{F}_T)$$

In particular $X_T \in L^1$ and $E(X_0) = E(X_T) = E(X_\infty)$.

<u>Rk:</u> (X_n) UI $\Leftrightarrow$ $X_n = E(X_\infty \mid \mathcal{F}_n)$ a.s

↑

we can replace n by a random time.

<u>Proof</u>. We first show $X_T \in L^1$

$$E(|X_T|) \overset{\text{Fubini}}{=} \sum_{n\in\mathbb{N}\cup\{\infty\}} E(|X_n| \mathbb{1}_{T=n})$$

$$= \sum_{n\in\mathbb{N}\cup\{\infty\}} E(|E(X_\infty|\mathcal{F}_n)| \mathbb{1}_{T=n})$$

$$\leq \sum_{n\in\mathbb{N}\cup\{\infty\}} E(E(|X_\infty| \mid \mathcal{F}_n) \mathbb{1}_{\underbrace{T=n}_{\in \mathcal{F}_n}})$$

$$= \sum_{n\in\mathbb{N}\cup\{\infty\}} E(|X_\infty| \mathbb{1}_{T=n}) \overset{\text{Fub}}{=} E(|X_\infty|).$$

- $X_T \in L^1$ X_T is $\mathcal{F}_T$-meas.

- Let $A \in \mathcal{F}_T$

use $X_T \in L^1$

$$E(X_T 1_A) \overset{\text{Fub.}}{=} \sum_n E(X_n 1_A 1_{T=n})$$

$$= \sum_n E\Big(E(X_\infty | \mathcal{F}_n)\, 1_{\underbrace{A \cap \{T=n\}}_{\in \mathcal{F}_n}}\Big)$$

$$= \sum_n E\left(X_\infty 1_{A \cap \{T=n\}}\right)$$

$X_\infty \in L^1$

$$\overset{\text{Fub}}{=} E(X_\infty 1_A).$$

This concludes $X_T = E(X_\infty | \mathcal{F}_T)$.

Taking the expectation gives $E(X_T) = E(X_\infty)$

and, since $X_0 = E(X_\infty | \mathcal{F}_0)$ we also have $E(X_0) = E(X_\infty)$. ∎

<u>Corollary</u>. Let $(X_n)_{n \geq 0}$ UI martingale, S, T stopping times s.t. $S \leq T$. We have

$$E(X_T | \mathcal{F}_S) = X_S \text{ a.s.}$$

<u>Rk</u>: generalization of $\forall p \geq n$ $E(X_p | \mathcal{F}_n) = X_n$ a.s to random times.

Pf: We have $\mathcal{F}_S \subset \mathcal{F}_T$. Hence by the tower property

$$E(X_T \mid \mathcal{F}_S) = E(E(X_\infty \mid \mathcal{F}_T) \mid \mathcal{F}_S)$$

$$= E(X_\infty \mid \mathcal{F}_S)$$

$$= X_S \quad \text{a.s.} \qquad \blacksquare$$