CHAPTER IS :

L^p MARTINGALES

Goals:
- maximal inequalities
- convergence Thms for L^p-martingales.

Setup:
- $(\Omega, \mathcal{F}, P)$ fixed probability space.
- $(\mathcal{F}_n)_{n \geq 0}$ filtration.
- $L^p = \{ X : \Omega \to \mathbb{R}$ r.v. s.t. $E(|X|^p) < \infty \}$, $p > 1$.

1 AN OPTIONAL STOPPING RESULT.

Lemma (an optional stopping result for submart.).

Let $(X_n)_{n \geq 0}$ be a submartingale, T a stopping time.
For every $n \in \mathbb{N}$, we have

$$\boxed{E(X_0) \leq E(X_{n \wedge T}) \leq E(X_n)}.$$

Proof. • $E(X_0) \leq E(X_{n\wedge T})$ holds because $(X_{n\wedge T})_{n\geq 0}$ submart.

• For $n \geq 0$, we have

$$(X_n - X_{n\wedge T}) = \sum_{k=1}^{n} 1_{k>T} \cdot (X_k - X_{k-1})$$

$$= (H \cdot X)_n ,$$

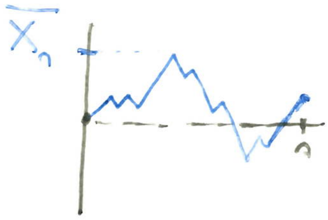
where $H_n = 1_{T<n}$ $\mathcal{F}_{n-1}$ - meas., bounded.

Hence $(X_n - X_{n\wedge T})$ submartingale, and we have

$$E(X_n - X_{n\wedge T}) \geq E(X_0 - X_{0\wedge T}) = 0 \qquad \blacksquare$$

2 DOOB MAXIMAL INEQUALITY.

Notation: $\overline{X_n} = \max_{0\leq k\leq n} X_k^+$



Thm (Doob maximal inequality).

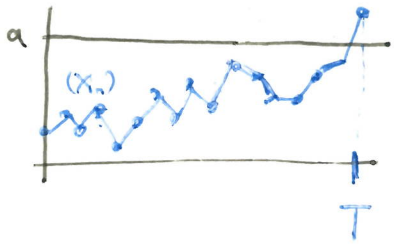
Let $(X_n)_{n\geq 0}$ be a submartingale. For every $n \geq 0$ for every $a > 0$, we have

$$P(\overline{X_n} \geq a) \leq \frac{1}{a} E(X_n 1_{\overline{X_n}\geq a}) \leq \frac{1}{a} E(X_n^+).$$

" X_n^+ controls the max $\overline{X_n}$ "

Rk: • Markov inequality would give $P(\overline{X}_n \geq a) \leq \frac{1}{a} E(\overline{X}_n)$.

Proof: Let $T = \min\{n \in \mathbb{N} : X_n \geq a\}$ ($a > 0$ fixed). Let $n \geq 0$.



Rk: $T \leq n \iff \overline{X}_n \geq a$

We have
$$E(X_{T\wedge n}) = E(X_T 1_{T\leq n}) + E(X_n 1_{T > n})$$
$$E(X_n) = E(X_n 1_{T\leq n}) + E(X_n 1_{T > n})$$

Since $E(X_{T\wedge n}) \leq E(X_n)$ (lemma), we get
$$E(\underbrace{X_T}_{\geq a} 1_{T\leq n}) \leq E(X_n 1_{T\leq n})$$

Hence $a\, P(\overline{X}_n \geq a) \leq E(X_n 1_{\overline{X}_n \geq a})$.

For the second inequality, use $X_n 1_A \leq X_n^+ 1_A \leq X_n^+$ for every A ∎

3 APPLICATIONS TO MARTINGALES

Reminder: $\left.\begin{array}{l}\varphi \text{ convex} \\ (X_n) \text{ mart}\end{array}\right\} \Rightarrow (\varphi(X_n))$ submart.

↳ we can apply Doob max inequality to $(\varphi(X_n))$.

Not.: For $n \geq 0$, write $X_n^* = \max_{0 \leq k \leq n} |X_k|$.

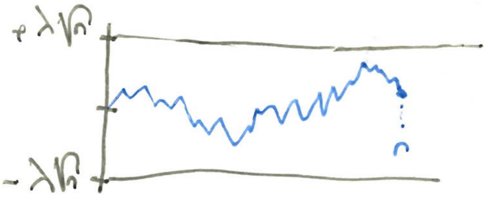
Prop. Let $(X_n)_{n \geq 0}$ be a martingale. For every $n \in \mathbb{N}$, for every $a > 0$, we have

$$P(X_n^* \geq a) \leq \frac{1}{a} E(|X_n|).$$

Pf. $(|X_n|)_{n \geq 0}$ is a non negative submartingale. Hence we can apply Doob maximal inequality.

Application to SRW

$Z_1, Z_2, \ldots$ iid $\mathcal{U}(\{-1, +1\})$. $X_n = Z_1 + \ldots + Z_n$



(X_n^2) submartingale. For all $\lambda > 0$

$$P\left(\max_{i \leq n} |X_i| \geq \lambda\sqrt{n}\right) = P\left(\max_{i \leq n} X_i^2 \geq \lambda^2 n\right)$$

$$\overset{\text{Doob}}{\leq} \frac{E(X_n^2)}{\lambda^2 n}$$

$$= \frac{1}{\lambda^2}$$

4 DOOB L^p INEQUALITY.

Thm: Let $(X_n)_{n\geq 0}$ be a submartingale. For every $p\in(1,+\infty)$,

$$E\left(\overline{X}_n^{\,p}\right) \leq \left(\frac{p}{p-1}\right)^p E\left((X_n^+)^p\right)$$

Rk: • $E((X_n^+)^p) \underset{\text{trivial}}{\leq} E(\overline{X}_n^{\,p}) \underset{\text{Doob } L^p}{\leq} C(p)\, E((X_n^+)^p)$

• in the deterministic case, if $(u_n) \nearrow$ we have

$$u_n = \max_{i\leq n} u_i$$

Exercise. (generalization of Tailsum).

Let Z be a non negative r.v. ($Z\geq 0$ a.s). $p\in[1,\infty)$

$$E(Z^p) = p\cdot\int_0^\infty t^{p-1}\, P(Z\geq t)\,dt$$

Reminder (Hölder). Let $p,q\in(1,\infty)$ s.t. $\frac{1}{p}+\frac{1}{q}=1$

For every $Y, Z \geq 0$ r.vs

$$E(YZ) \leq E(Y^p)^{\frac{1}{p}}\, E(Z^q)^{\frac{1}{q}}.$$

Proof of Thm: Let $n \geq 0$. Set $Y = X_n^+$, $Z = \overline{X}_n$.

$$E(Z^p) = p \int_0^\infty t^{p-1} P(Z \geq t)\, dt$$

$$\overset{\text{Doob}}{\leq} p \int_0^\infty t^{p-2} E(Y 1_{Z \geq t})\, dt$$

$$\overset{\text{Fubini}}{=} p\, E\Big(Y \cdot \underbrace{\Big(\int_0^\infty t^{p-2} 1_{Z \geq t}\, dt\Big)}_{= \int_0^Z t^{p-2} dt \,=\, \frac{1}{p-1} Z^{p-1}}\Big)$$

$$= \frac{p}{p-1} E(Y Z^{p-1})$$

Let $q \in (1, \infty)$ s.t. $\frac{1}{p} + \frac{1}{q} = 1$.

<u>NB</u>: $\frac{1}{p} + \frac{1}{q} = 1 \Leftrightarrow p + q = pq \Leftrightarrow (p-1) q = p$

By Hölder inequality, we get

$$E(Z^p) \leq \left(\frac{p}{p-1}\right) E(Y^p)^{\frac{1}{p}} E\left(Z^p\right)^{\frac{1}{q}}$$

Dividing by $E(Z^p)^{\frac{1}{q}}$ we obtain

$$E(Z^p)^{\frac{1}{p}} \leq \left(\frac{p}{p-1}\right) E(Y^p)^{\frac{1}{p}}$$

∎

Rk: $\left(\frac{p}{1-p}\right)^p \underset{p \searrow 1}{\longrightarrow} \infty$.

A L^1-version requires a log:

$$E(\bar{X}_n) \leq \left(1-\frac{1}{e}\right)^{-1} E(X_n^+ \log X_n^+)$$ (see Exercises).

Corollary. Let $(X_n)_{n\geq 0}$ be a martingale, $p \in (1,\infty)$.

For every $n \geq 1$, we have

$$E((X_n^*)^p) \leq \left(\frac{p}{p-1}\right)^p E(|X_n|^p).$$

Proof. $(|X_n|)$ is a submartingale.

Application $Z_1, Z_2, \ldots$ iid $\mathcal{U}(\{+1,-1\})$. $X_n = Z_1 + \cdots + Z_n$

$$E\left(\max_{i\leq n} S_i^2\right) \leq 4\, E(S_n^2) = 4n.$$

5 MARTINGALES BOUNDED IN L^p.

Rk. If a martingale $(X_n)_{n\geq 0}$ is bounded in L^p, $p>1$

(ie $\sup_{n\geq 0} E(|X_n|^p) < \infty$), then in particular

it is UI: $X_n \xrightarrow{a.s., L^1} X_\infty$ and $\forall n$ $X_n = E(X_\infty | \mathcal{F}_n)$

Not. $X^*_\infty = \sup_{n\geq 0} |X_n| = \lim_{n\to\infty} \uparrow X^*_n$.

Thm: Let $p \in (1,\infty)$. Let $(X_n)_{n\geq 0}$ be a martingale s.t.

$$\sup E(|X_n|^p) < \infty .$$

Then $(X_n)_{n\geq 0}$ cv a.s. and in L^p to a r.v. $X_\infty \in L^p$.

Furthermore

$$E(|X^*_\infty|^p) \leq \left(\frac{p}{p-1}\right)^p E(|X_\infty|^p)$$

"Doob L^p at time $n=+\infty$"

Rk: • The thm does not hold in general for $p=1$.

$\sup E(|X_n|) < \infty \Rightarrow$ a.s cv but not L^1 cv in general.

• $E(|X_\infty|^p) \underset{\substack{\uparrow \\ L^p\text{-cv.}}}{=} \lim_{n\to\infty} E(|X_n|^p) \underset{\substack{\uparrow \\ |X_n|^p \text{ submart.}}}{=} \sup_{n\geq 0} E(|X_n|^p)$.

Proof. Since (X_n) bounded in L^1, we have

$$X_n \xrightarrow{a.s.} X_\infty \text{ where } X_\infty \in L^1$$

By Doob L^p-ineq., for every $n \geq 0$ we have

$$E(|X_n^*|^p) \leq \left(\frac{p}{p-1}\right)^p \underbrace{E(|X_n|^p)}_{\leq C := \sup_{n \geq 0} E(|X_n|^p)} \qquad (*)$$

Monotone cv $\downarrow$

$$E(|X_\infty^*|^p)$$

Hence $(X_\infty^*)^p \in L^1$.

For every $n \geq 0$ $|X_n - X_\infty| \leq |X_n| + |X_\infty| \leq 2|X_\infty^*|$ a.s.

Hence, by dominated cv. $X_n \xrightarrow{L^p} X_\infty$.

In particular, we can take the limit in the inequality $(*)$ to get

$$E(|X_\infty^*|^p) \leq \left(\frac{p}{p-1}\right)^p E(|X_\infty|^p.$$

CHAPTER 16.

BACKWARD MARTINGALES

<u>Setup</u> : $(\Omega, \mathcal{F}, P)$ proba. space.

1 BACKWARD FILTRATION.

<u>Def.</u> A <u>backward filtration</u> is a sequence $(\mathcal{F}_n)_{n \in \mathbb{Z}_-}$ of σ-algebras $(\mathcal{F}_n \subset \mathcal{F})$ s.t.

$$\forall n \in \mathbb{Z}_- \quad \mathcal{F}_{n-1} \subset \mathcal{F}_n$$

$(\mathbb{Z}_- = \{0, -1, -2, \dots\})$

" gain of information when time increases "

Filtration :

$$\overset{0}{\bullet} \quad \overset{1}{\bullet} \quad \overset{2}{\bullet} \quad \cdots$$

$$\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \cdots \quad \cdots \mathcal{F}_\infty$$

Backward filtration $\mathcal{F}_{-\infty}$

$$\cdots \overset{-2}{\bullet} \quad \overset{-1}{\bullet} \quad \overset{0}{\bullet}$$

$$\mathcal{F}_{-2} \subset \mathcal{F}_{-1} \subset \mathcal{F}_0$$

" loss of information when we go back in time "

<u>Notation</u>: $\boxed{\mathcal{F}_{-\infty} := \bigcap_{n \in \mathbb{Z}_-} \mathcal{F}_n}$

<u>Example</u>. $(X_n)_{n \leq 0}$ r.v. $\quad \mathcal{F}_n = \sigma(X_n, X_{n-1}, X_{n-2}, \dots)$ back. filtr..

In the rest of the chapter, we fix a backward filtration $(\mathcal{F}_n)_{n \leq 0}$:

2 BACKWARD MARTINGALES

Def. A sequence of r.v.s $(X_n)_{n \leq 0}$ is a backward martingale if

- $\forall n \leq 0$ X_n is $\mathcal{F}_n$-measurable and $X_n \in L^1$
- $\forall n \leq 0$ $E(X_n \mid \mathcal{F}_{n-1}) = X_{n-1}$ a.s.

Ex 1 Let $X \in L^1$. $X_n = E(X \mid \mathcal{F}_n)$ defines a backward martingale.

$$\Big(\forall n \leq 0 \quad E(X_n \mid \mathcal{F}_{n-1}) = E(E(X \mid \mathcal{F}_n) \mid \mathcal{F}_{n-1}) \underset{\mathcal{F}_{n-1} \subset \mathcal{F}_n}{=} E(X \mid \mathcal{F}_{n-1})\Big)$$

Ex 2 Let $Z_{-1}, Z_{-2} \ldots$ iid $Ber(\frac{1}{2})$.

$$\mathcal{F}_n = \sigma(Z_{n-1}, Z_{n-2}, \ldots)$$

$$X_0 := \sum_{n \leq -1} Z_n 2^n = Z_{-1} \cdot \frac{1}{2} + Z_{-2} \cdot \frac{1}{4} + Z_{-3} \cdot \frac{1}{8} + \ldots$$

"replace by average"

$$X_{-1} = \frac{1}{4} + \sum_{k \leq -2} Z_k 2^k = \frac{1}{4} + Z_{-2} \cdot \frac{1}{4} + Z_{-3} \cdot \frac{1}{8} + \ldots$$

$$X_{-2} = \frac{1}{4} + \frac{1}{8} + \sum_{k \leq -3} Z_k 2^k = \frac{1}{4} + \frac{1}{8} + Z_{-3} \cdot \frac{1}{8} \ldots$$

$\vdots$

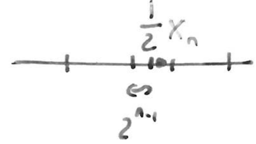
$$X_n = \sum_{n \leq k \leq -1} \frac{1}{2} \cdot 2^k + \sum_{k \leq n} Z_k 2^k$$

<u>Rk</u>: $X_0 \sim \mathcal{U}([0,1])$

$X_1 \sim \mathcal{U}([\frac{1}{4}, \frac{3}{4}])$

$\vdots$

$X_n \sim \mathcal{U}\left([\frac{1}{2} - 2^{n-1}, \frac{1}{2} + 2^{n-1}]\right)$



<u>Exercise</u>: check that $(X_n)_{n \leq 0}$ is a backward martingale.

<u>Rk</u>: Let $Z_1, Z_2, \dots$ iid $\mathrm{Ber}(\frac{1}{2})$ $\mathcal{F}_n = \sigma(Z_1, \dots, Z_n)$ filtration

$X_n = Z_1 \times \frac{1}{2} + \dots + Z_n \times \frac{1}{2^n}$ (forward) - martingale.

<u>Prop</u>. Let (X_n) be a backward martingale. Then for every $m, n \leq 0$ s.t. $m \leq n$, we have

$$\boxed{E(X_n \mid \mathcal{F}_m) = X_m}$$

<u>Proof</u>. Fix $m \leq 0$. Let $k \in \{1, \dots, m\}$

$$E(X_{m+k} \mid \mathcal{F}_m) \underset{\mathcal{F}_m \subset \mathcal{F}_{m+k-1}}{=} E\left(E(X_{m+k} \mid \mathcal{F}_{m+k-1}) \mid \mathcal{F}_m\right)$$

$$= E(X_{m+k-1} \mid \mathcal{F}_m) \quad \text{a.s.}$$

And the result follows by finite induction.

<u>Rk</u>: In particular $\forall\, n \leq 0$ $\boxed{X_n = E(X_0 \mid \mathcal{F}_n)}$

and $\boxed{E(X_n) = E(X_0)}$

3 CONVERGENCE THM

Thm. Let $(X_n)_{n \leq 0}$ be a backward martingale.

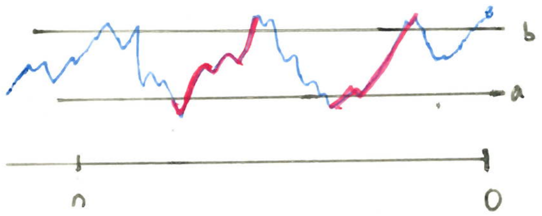
$$X_n \xrightarrow[n \to -\infty]{} E(X_0 \mid \mathcal{F}_{-\infty}) \text{ a.s and in } L^1.$$

Rk: no hypothese because a back. martingale is automatically "closed". In particular, since $X_n = E(X_0 \mid \mathcal{F}_n)$, the sequence is automatically U.I.

Pf. Sketch: a.s cv.

For $a < b$ $a, b \in \mathbb{Q}$, define

$$N_n(a,b) = |\{\text{upcrossings of } (X_n, \dots, X_0)\}|$$



Doob upcrossing inequality:

$$\forall \underset{a,b \in \mathbb{Q}}{a<b} \quad \underbrace{E(N_n(a,b))}_{\downarrow \text{ monotone cv}} \leq \frac{1}{b-a} E\left((X_0 - a)^-\right)$$

$$E(N_{-\infty}(a,b))$$

"total number of upcrossings"

Hence a.s. we have

$$\forall a<b, \; a,b\in\mathbb{Q} \quad N_{-\infty}(a,b) < \infty,$$

which deterministically implies the existence of $X_{-\infty}$ s.t

$$X_n \xrightarrow[n\to\infty]{} X_{-\infty} \quad a.s.$$

<u>L^1 cv and identification of the limit.</u>

Since $X_n = E(X_0 \mid \mathcal{F}_n)$, the sequence (X_n) is UI. Since it cv a.s, it also cv in L^1.

We have $X_n \xrightarrow[n\to-\infty]{L^1} X_{-\infty}$. (NB: $X_{-\infty}$ is $\mathcal{F}_{-\infty}$-meas.)

Let $A\in\mathcal{F}_{\infty}$. For every $n\leq 0$, $A\in\mathcal{F}_n$, hence

$$E(X_0 1_A) = E\left(E(X_0\mid\mathcal{F}_n)\,1_A\right)$$
$$= \underbrace{E(X_n 1_A)}$$
$$\xrightarrow[n\to-\infty]{} E(X_{\infty} 1_A) \quad \text{(because } X_n \to X_{-\infty}\text{)}$$

Therefore $X_{-\infty} = E(X_0 \mid \mathcal{F}_{\infty})$ a.s $\blacksquare$

Application: Let $Z\in L^1$

$$E(Z\mid\mathcal{F}_n) \xrightarrow{L^1, a.s} E(Z\mid\mathcal{F}_{\infty})$$

<u>Appli. 2</u> : $X_n = \frac{1}{2} - 2^{-n-1} + \sum_{k \le n-1} Z_k 2^k$;

$$X_n \xrightarrow[n \to -\infty]{} \frac{1}{2} \quad \text{a.s. and in } L^1$$

<u>Pf. 1</u> : $\left| \sum_{k \le n-1} Z_k 2^k \right| \le \sum_{k \le n-1} 2^k = 2^n \xrightarrow[n \to -\infty]{} 0$.

<u>Pf 2</u> : $X_{-\infty}$ is $\mathcal{F}_{-\infty}$-meas. Since $\mathcal{F}_{-\infty}$ is P-trivial

$(\forall A \in \mathcal{F}_{-\infty} \quad P(A \in \{0,1\}))$ $X_{-\infty} = C$ a.s.

where C = Cte. and

$$C = E(X_{-\infty}) = E(X_0) = \frac{1}{2}.$$

4 LAW OF LARGE NUMBERS REVISITED.

Let $Z_1, Z_2, \dots$ iid $Z_i \in L^1$. For $k \geq 0$ $S_k = Z_1 + \dots + Z_k$

We give a new proof of $\frac{S_k}{k} \xrightarrow[k\to\infty]{} E(Z_1)$ a.s. and in L^1.

Define for $k \geq 0$

$$\mathcal{F}_{-k} = \sigma(S_{k+1}, Z_{k+2}, Z_{k+3}, \dots)$$

$$X_{-k} = \frac{S_{k+1}}{k+1}$$

$\mathcal{F}_0 = \sigma(Z_1, Z_2, \dots)$ $\qquad X_0 = Z_1$

$\mathcal{F}_{-1} = \sigma(S_2, Z_2, \dots)$ $\qquad X_{-1} = \frac{Z_1 + Z_2}{2}$

$\mathcal{F}_{-2} = \sigma(S_3, Z_3, \dots)$ $\qquad X_{-2} = \frac{Z_1 + Z_2 + Z_3}{3}$

$\vdots$

Rk: $(\mathcal{F}_n)_{n \leq 0}$ is a backward filtration; Indeed, for all $k \geq 0$

$$\mathcal{F}_{-k-1} = \sigma(S_{k+2}, Z_{k+3}, \dots)$$
$$\subset \sigma(S_{k+1}, Z_{k+2}, Z_{k+3}, \dots) = \mathcal{F}_{-k}$$

because $S_{k+2} = S_{k+1} + Z_{k+2}$

Prop. $(X_n)_{n \leq 0}$ is a $(\mathcal{F}_n)_{n\leq 0}$-backward martingale.

Proof We show that

$$\forall n \leq 0 \quad X_n = E(Z_1 \mid \mathcal{F}_n).$$

Fix $k \geq 1$. We prove

$$\frac{S_k}{k} = E(Z_1 \mid S_k, Z_{k+1}, \dots)$$

Since $Z_{k+1}, Z_{k+2}, \dots$ are indep of (Z_1, S_k)

it suffices to show.

$$\frac{S_k}{k} = E(Z_1 \mid S_k). \ .$$

By symmetry, we have

$$\forall i \leq k \quad E(Z_1 \mid S_k) = E(Z_i \mid S_k)$$

(To prove this statement, consider $A \in \sigma(S_k)$

ie $1_A = \varphi(S_k) = \varphi(Z_1 + \dots + Z_k)$, for some φ meas. bounded.

Since $(Z_1, S_k) \overset{(d)}{=} (Z_i, S_k)$, we have

$$E(Z_1 \varphi(Z_1 + \dots + Z_n)) = E(Z_i \varphi(Z_1 + \dots + Z_n))$$

ie $$E(Z_1 1_A) = E(Z_i 1_A))$$

Hence $k\, E(Z_1 | S_k) = E(Z_1 | S_k) + \dots + E(Z_k | S_k)$

$$= E(S_k | S_k)$$

$$= S_k \qquad \blacksquare$$

Hence $\exists X_{-\infty} \in L^1$ s.t. $X_{n'} \xrightarrow[n' \to -\infty]{} X_{-\infty}$ a.s. and in L^1

By Kolmogorov 0-1 law,

$$X_{-\infty} = \lim_{k \to \infty} \frac{S_k}{k} \quad \text{is constant a.s.}$$

(because $\forall a \ \{\lim_{k \to \infty} \frac{S_k}{k} \leq a\} \in \mathcal{T} = \bigcap_{k \geq 0} \sigma(Z_k, Z_{k+1}, \dots)$.)

$\exists c$ s.t. $X_{-\infty} = c$ a.s.

But $X_{-\infty} = E(Z_1 | \mathcal{G}_{-\infty})$ so we must have

$$\boxed{E(Z_1) = c}$$

This concludes

$$\frac{Z_1 + \dots + Z_k}{k} \xrightarrow[k \to \infty]{} E(Z_1) \quad \text{a.s and in } L^1 \qquad \blacksquare$$