CHAPTER 16.

BACKWARD MARTINGALES

Setup: $(\Omega, \mathcal{F}, P)$ proba. space.

I BACKWARD FILTRATION.

Def. A backward filtration is a sequence $(\mathcal{F}_n)_{n \in \mathbb{Z}_-}$ of σ-algebras ($\mathcal{F}_n \subset \mathcal{F}$) s.t.

$$\forall n \in \mathbb{Z}_- \quad \mathcal{F}_{n-1} \subset \mathcal{F}_n$$

$(\mathbb{Z}_- = \{0, -1, -2, \dots\})$

" gain of information when time increases"

Filtration:

$$\begin{array}{ccccc} 0 & 1 & 2 & & \\ \bullet & \bullet & \bullet & \cdots & \\ \mathcal{F}_0 \subset & \mathcal{F}_1 \subset & \mathcal{F}_2 & \cdots & \cdots \mathcal{F}_\infty \end{array}$$

Backward filtration

$$\begin{array}{ccccc} & & -2 & -1 & 0 \\ \mathcal{F}_{-\infty} & \cdots & \bullet & \bullet & \bullet \\ & & \mathcal{F}_{-2} \subset & \mathcal{F}_{-1} \subset & \mathcal{F}_0 \end{array}$$

" loss of information when we go back in time"

Notation: $\boxed{\mathcal{F}_{-\infty} := \bigcap_{n \in \mathbb{Z}_-} \mathcal{F}_n}$

Example. $(X_n)_{n \leq 0}$ r.v. $\mathcal{F}_n = \sigma(X_n, X_{n-1}, X_{n-2}, \dots)$ back. filtr..

In the rest of the chapter, we fix a backward filtration $(\mathcal{F}_n)_{n\leq 0}$.

2 BACKWARD MARTINGALES

Def. A sequence of r.v.s $(X_n)_{n\leq 0}$ is a backward martingale if

- $\forall n \leq 0 \quad X_n$ is $\mathcal{F}_n$-measurable and $X_n \in L^1$
- $\forall n \leq 0 \quad E(X_n \mid \mathcal{F}_{n-1}) = X_{n-1}$ a.s.

Ex 1 Let $X \in L^1$. $X_n = E(X \mid \mathcal{F}_n)$ defines a backward martingale.

$$\Big(\forall n \leq 0 \quad E(X_n \mid \mathcal{F}_{n-1}) = E(E(X \mid \mathcal{F}_n) \mid \mathcal{F}_{n-1}) \underset{\mathcal{F}_{n-1} \subset \mathcal{F}_n}{=} E(X \mid \mathcal{F}_{n-1})\Big)$$

Ex 2 Let $Z_{-1}, Z_{-2} \dots$ iid $\mathrm{Ber}(\frac{1}{2})$.

$$\mathcal{F}_n = \sigma(Z_{n-1}, Z_{n-2}, \dots)$$

$$X_0 := \sum_{n\leq -1} Z_n 2^n = Z_{-1}\cdot\frac{1}{2} + Z_{-2}\cdot\frac{1}{4} + Z_{-3}\cdot\frac{1}{8} + \dots$$

"replace by average"

$$X_{-1} = \frac{1}{4} + \sum_{k\leq -2} Z_k 2^k = \frac{1}{4} + Z_{-2}\cdot\frac{1}{4} + Z_{-3}\cdot\frac{1}{8} + \dots$$

$$X_{-2} = \frac{1}{4} + \frac{1}{8} + \sum_{k\leq -3} Z_k 2^k = \frac{1}{4} + \frac{1}{8} + Z_{-3}\cdot\frac{1}{8} \dots$$

$\vdots$

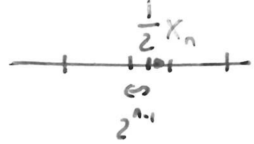
$$X_n = \sum_{n\leq k\leq -1} \frac{1}{2}\cdot 2^k + \sum_{k\leq n} Z_k 2^k$$

<u>Rk:</u> $X_0 \sim \mathcal{U}([0,1])$

$X_1 \sim \mathcal{U}([\frac{1}{4}, \frac{3}{4}])$

$\vdots$

$X_n \sim \mathcal{U}\left(\left[\frac{1}{2} - 2^{n-1}, \frac{1}{2} + 2^{n-1}\right]\right)$



<u>Exercise:</u> check that $(X_n)_{n \leq 0}$ is a backward martingale.

<u>Rk:</u> Let $Z_1, Z_2, \ldots$ iid $\mathrm{Ber}(\frac{1}{2})$ $\mathcal{F}_n = \sigma(Z_1, \ldots, Z_n)$ filtration

$X_n = Z_1 \times \frac{1}{2} + \cdots + Z_n \times \frac{1}{2^n}$ (forward) - martingale.

<u>Prop.</u> Let (X_n) be a backward martingale. Then for every $m, n \leq 0$ s.t. $m \leq n$, we have

$$\boxed{E(X_n \mid \mathcal{F}_m) = X_m}$$

<u>Proof.</u> Fix $m \leq 0$. Let $k \in \{1, \ldots, m\}$

$$E(X_{m+k} \mid \mathcal{F}_m) \underset{\mathcal{F}_m \subset \mathcal{F}_{m+k-1}}{=} E(E(X_{m+k} \mid \mathcal{F}_{m+k-1}) \mid \mathcal{F}_m)$$

$$= E(X_{m+k-1} \mid \mathcal{F}_m) \quad \text{a.s.}$$

And the result follows by finite induction.

<u>Rk:</u> In particular $\forall\, n \leq 0$ $\boxed{X_n = E(X_0 \mid \mathcal{F}_n)}$

and $\boxed{E(X_n) = E(X_0)}$

3 CONVERGENCE THM

Thm. Let $(X_n)_{n \leq 0}$ be a backward martingale.

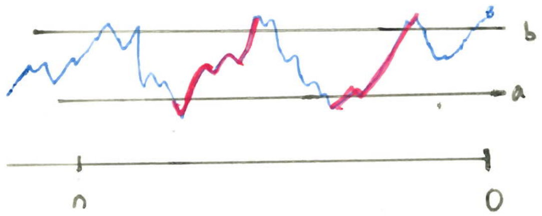
$$X_n \xrightarrow[n \to -\infty]{} E(X_0 \mid \mathcal{F}_{-\infty}) \text{ a.s and in } L^1.$$

Rk: no hypothese because a back. martingale is automatically "closed". In particular, since $X_n = E(X_0 \mid \mathcal{F}_n)$, the sequence is automatically U. I.

Pf. Sketch: a.s cv.

For $a < b$ $a, b \in \mathbb{Q}$, define

$$N_n(a,b) = |\{ \text{upcrossings of } (X_n, \dots, X_0) \}|$$



Doob upcrossing inequality:

$$\forall \underset{a,b \in \mathbb{Q}}{a < b} \quad \underbrace{E(N_n(a,b))}_{\downarrow \text{ monotone cv}} \leq \frac{1}{b-a} E((X_0 - a)^-)$$

$$E(N_{-\infty}(a,b))$$

"total number of upcrossings"

Hence a.s. we have

$$\forall \underset{a,b\in\mathbb{Q}}{a<b} \quad N_{-\infty}(a,b) < \infty ,$$

which deterministically implies the existence of $X_{-\infty}$ s.t

$$X_n \xrightarrow[n\to\infty]{} X_{-\infty} \quad a.s$$

L^1 cv and identification of the limit.

Since $X_n = E(X_0 \mid \mathcal{F}_n)$, the sequence (X_n) is UI. Since it cv a.s, it also cv in L^1.

We have $X_n \xrightarrow[n\to-\infty]{L^1} X_{-\infty}$. (NB: $X_{-\infty}$ is $\mathcal{F}_{-\infty}$-meas.)

Let $A \in \mathcal{F}_{\infty}$. For every $n \leq 0$, $A \in \mathcal{F}_n$, hence

$$E(X_0 1_A) = E\big(E(X_0 \mid \mathcal{F}_n)\, 1_A\big)$$
$$= \underbrace{E(X_n 1_A)}_{\xrightarrow[n\to-\infty]{} E(X_{\infty} 1_A)}$$

(because $X_n \to X_{-\infty}$)

Therefore $X_{-\infty} = E(X_0 \mid \mathcal{F}_{\infty})$ a.s $\square$

Application: Let $Z \in L^1$

$$E(Z \mid \mathcal{F}_n) \xrightarrow{L^1, a.s} E(Z \mid \mathcal{F}_{\infty})$$

Appli. 2 : $X_n = \frac{1}{2} - 2^{-n-1} + \sum_{k \leq n-1} Z_k 2^k$:

$$X_n \xrightarrow[n \to -\infty]{} \frac{1}{2} \quad \text{a.s. and in } L^1$$

Pf. 1 : $\left| \sum_{k \leq n-1} Z_k 2^k \right| \leq \sum_{k \leq n-1} 2^k = 2^n \xrightarrow[n \to -\infty]{} 0$.

Pf 2 : $X_{-\infty}$ is $\mathcal{F}_{-\infty}$-meas. Since $\mathcal{F}_{-\infty}$ is P-trivial

$(\forall A \in \mathcal{F}_{-\infty} \quad P(A \in \{0,1\}))$ $X_{-\infty} = C$ a.s.

where $C = $ Cte. and

$$C = E(X_{-\infty}) = E(X_0) = \frac{1}{2}.$$

4 LAW OF LARGE NUMBERS REVISITED.

Let $Z_1, Z_2, \dots$ iid $Z_i \in L^1$. For $k \geq 0$ $S_k = Z_1 + \dots + Z_k$

We give a new proof of $\frac{S_k}{k} \xrightarrow[k \to \infty]{} E(Z_1)$ a.s. and in L^1.

Define for $k \geq 0$

$$\mathcal{F}_{-k} = \sigma(S_{k+1}, Z_{k+2}, Z_{k+3}, \dots)$$

$$X_{-k} = \frac{S_{k+1}}{k+1}$$

$\mathcal{F}_0 = \sigma(Z_1, Z_2, \dots)$ $\qquad X_0 = Z_1$

$\mathcal{F}_{-1} = \sigma(S_2, Z_2, \dots)$ $\qquad X_{-1} = \frac{Z_1 + Z_2}{2}$

$\mathcal{F}_{-2} = \sigma(S_3, Z_3, \dots)$ $\qquad X_{-2} = \frac{Z_1 + Z_2 + Z_3}{3}$

$\vdots$

Rk: $(\mathcal{F}_n)_{n \leq 0}$ is a backward filtration; Indeed, for all $k \geq 0$

$$\mathcal{F}_{-k-1} = \sigma(S_{k+2}, Z_{k+3}, \dots)$$
$$\subset \sigma(S_{k+1}, Z_{k+2}, Z_{k+3}, \dots) = \mathcal{F}_{-k}$$

because $S_{k+2} = S_{k+1} + Z_{k+2}$

Prop. $(X_n)_{n\leq 0}$ is a $(\mathcal{F}_n)_{n\leq 0}$-backward martingale.

Proof We show that

$$\forall n \leq 0 \quad X_n = E(Z_1 \mid \mathcal{F}_n).$$

Fix $k \geq 1$. We prove

$$\frac{S_k}{k} = E(Z_1 \mid S_k, Z_{k+1}, \dots)$$

Since $Z_{k+1}, Z_{k+2}, \dots$ are indep of (Z_1, S_k) it suffices to show.

$$\frac{S_k}{k} = E(Z_1 \mid S_k).$$

By symmetry, we have

$$\forall i \leq k \quad E(Z_1 \mid S_k) = E(Z_i \mid S_k)$$

(To prove this statement, consider $A \in \sigma(S_k)$

ie $1_A = \varphi(S_k) = \varphi(Z_1 + \dots + Z_k)$, for some φ meas. bounded.

Since $(Z_1, S_k) \overset{(d)}{=} (Z_i, S_k)$, we have

$$E(Z_1 \varphi(Z_1 + \dots + Z_n)) = E(Z_i \varphi(Z_1 + \dots + Z_n))$$

ie $\quad E(Z_1 1_A) = E(Z_i 1_A)$)

Hence $k\, E(Z_1 \mid S_k) = E(Z_1 \mid S_k) + \dots + E(Z_k \mid S_k)$

$$= E(S_k \mid S_k)$$

$$= S_k \quad \blacksquare$$

Hence $\exists X_{-\infty} \in L^1$ s.t. $X_{n} \xrightarrow[n \to -\infty]{} X_{-\infty}$ a.s. and in L^1

By Kolmogorov 0-1 law,

$$X_{-\infty} = \lim_{k \to \infty} \frac{S_k}{k} \quad \text{is constant a.s.}$$

(because $\forall a \left\{ \lim_{k \to \infty} \frac{S_k}{k} \leq a \right\} \in \mathcal{T} = \bigcap_{k \geq 0} \sigma(Z_k, Z_{k+1}, \dots)$.)

$\exists c$ s.t. $X_{-\infty} = c$ a.s.

But $X_{-\infty} = E(Z_1 \mid \mathcal{G}_{-\infty})$ so we must have

$$\boxed{E(Z_1) = c}$$

This concludes

$$\frac{Z_1 + \dots + Z_k}{k} \xrightarrow[k \to \infty]{} E(Z_1) \quad \text{a.s and in } L^1 \quad \blacksquare$$