CHAPTER 2

LAW OF LARGE NUMBERS.

Goals:
- precise understanding of the statement
- complete proof of L^1 LLN (Law of large numbers)
- strategy vs tactics/tricks in a mathematical proof.
- methods: truncation / use of subsequence.

Setup:
- $(\Omega, \mathcal{F}, P)$ probability space (fixed)
- $L^p = \{ X : \Omega \to \mathbb{R} \text{ measurable } E(|X|^p) < \infty \}$, $p \geq 1$.

  ↳ vector space ($X, Y \in L^p \Rightarrow X + Y \in L^p$)

1 STATEMENT

Thm (LLN)

Let $X_1, X_2, \dots \in L^1$, pairwise independent, identically distributed. Write $m = E(X_1)$.

$$\boxed{\lim_{n \to \infty} \frac{X_1 + \dots + X_n}{n} = m \quad \text{a.s.}}$$

Is it intuitive?

Consider X_n iid $\quad X_n \sim \mathrm{Ber}\left(\frac{1}{2}\right)$ "coin flip"

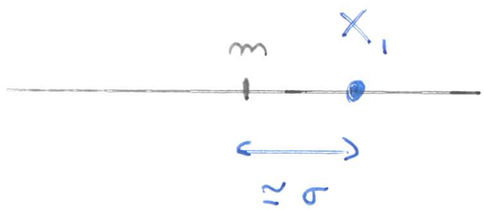
$$\frac{X_1 + \cdots + X_n}{n} = \frac{\text{"number of heads in } n \text{ throws"}}{n}$$

$$\to \frac{1}{2} = E(X_1)$$

A theoretical explanation: "averaging reduces the variance".

Assume $X_1 \in L^2$.

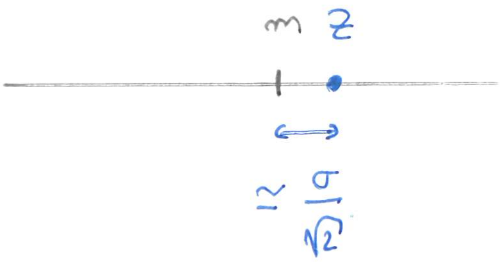
$$\begin{cases} m = E(X_1) & \text{"typical value"} \\ \sigma = \sqrt{\mathrm{Var}(X_1)} & \text{"typical distance from } m\text{"} \end{cases}$$



Consider $Z = \dfrac{X_1 + X_2}{2}$ where X_2 is an independent copy of X_1

$$E(Z) = \frac{1}{2}\left(E(X_1) + E(X_2)\right) = m$$ "same typical value as X_1"

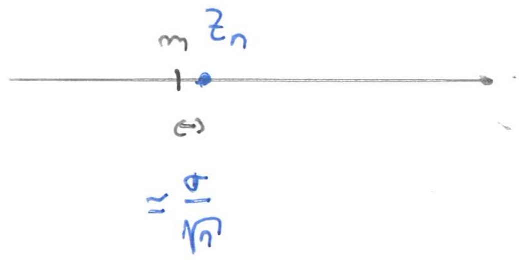
$$\mathrm{Var}(Z) = \frac{1}{4}\left(\mathrm{Var}(X_1) + \mathrm{Var}(X_2)\right) = \frac{1}{2}\mathrm{Var}(X_1)$$ "typical distance from m is smaller"



for $n \geq 1$, $Z_n = \dfrac{X_1 + \dots + X_n}{n}$

we have $E(Z_n) = m$

$$\sigma_n := \sqrt{\operatorname{Var}(Z_n)} = \frac{1}{\sqrt{n}}$$



If $X_1 \in L^1 \setminus L^2$ the "typical distance" between m and Z_n may be more than $\frac{1}{\sqrt{n}}$ but stays $o(1)$.

Applications:

see WUS.math (Spring 2024):
Monte-Carlo methods to approximate integrals, proof of Stone Weierstrass density theorem, ...

Strategy of the proof in this chapter: $S_n = X_1 + \dots + X_n$

Idea 1: use the criterion $\sum P\left(\left|\frac{S_n}{n} - m\right| > \varepsilon\right) < \infty$.

Idea 2: use Markov / Tchebitchev to estimate

$$P\left(\left|\frac{S_n}{n} - m\right| > \varepsilon\right).$$

2 L^4 - PROOF

Additional assumptions : $X_1 \in L^4$. and $X_1, X_2, \dots$ are iid.

Define : for every n $\quad Y_n := X_n - m \qquad (E(Y_n) = 0)$

$$S_n = X_1 + \dots + X_n$$

Let $\varepsilon > 0$.. For every $n \geq 1$ we have

$$P\left(\left| \frac{S_n}{n} - m \right| \geq \varepsilon \right) = P\left(\left| \frac{Y_1 + \dots + Y_n}{n} \right| \geq \varepsilon \right)$$

$$\overset{\text{Markov}}{\leq} \frac{1}{\varepsilon^4 n^4} E\left((Y_1 + \dots + Y_n)^4 \right)$$

$$= \frac{1}{\varepsilon^4 n^4} \sum_{1 \leq i,j,k,l \leq n} E(Y_i Y_j Y_k Y_l)$$

As soon as a factor X_α appears exactly once in the product $X_i X_j X_k X_l$, then independence implies that the expectation of this term vanishes. The only non-vanishing terms are of the form

$E(X_i^2 X_j^2) \qquad 1 \leq i, j \leq n$.

Therefore, for every $\varepsilon > 0$

$$P\left(\left|\frac{S_n}{n} - m\right| \geq \varepsilon\right) \leq \frac{1}{\varepsilon^4 n^4} \sum_{1 \leq i,j \leq n} \underbrace{E(Y_i^2 Y_j^2)}_{\overset{CS}{\leq} \sqrt{E(Y_i^4)}\sqrt{E(Y_j^4)} = E(Y_1^4)}$$

$$\leq \frac{3\, E(Y_1^4)}{\varepsilon^4} \times \frac{1}{n^2}.$$

Since $\sum_{n \geq 1} P\left(\left|\frac{S_n}{n} - m\right| \geq \varepsilon\right) < \infty$, we have

$$\lim_{n \to \infty} \frac{S_n}{n} = m \quad \text{a.s.} \qquad \blacksquare$$

3 L^2 - PROOF

Assumption: $X_1, X_2, \ldots \in L^2$, ident. distributed, pairwise independent.

For every $\varepsilon > 0$, $n \geq 1$, we have

$$P\left(\left|\frac{S_n}{n} - m\right| \geq \varepsilon\right) \leq \frac{1}{\varepsilon^2} \mathrm{Var}\left(\frac{S_n}{n}\right) = \frac{1}{\varepsilon^2 n} \mathrm{Var}(X_1)$$

Chebyshev inequality.

pairwise indep.

Pb: $\sum_{n\geq 1} \frac{1}{n} = +\infty$.

Consider subsequences.

Lemma.

Let $(u_n)_{n\geq 0}$ be a sequence of real numbers satisfying: ① $\forall\, 1 \leq m \leq n \quad 0 \leq m u_m \leq n u_n$,

② $\exists\, \ell \geq 0$ s.t.

$$\forall\, \alpha \in (1,\infty)\cap\mathbb{Q} \quad \lim_{i\to\infty} u_{\lfloor \alpha^i \rfloor} = \ell .$$

Then $\boxed{\lim_{n\to\infty} u_n = \ell.}$

Proof. Let $\alpha > 1$.

To every n, we associate $i = i(n)$ satisfying

$$\lfloor \alpha^i \rfloor \leq n \leq \lfloor \alpha^{i+1} \rfloor$$

By ①, we have

$$\forall n \geq 1 \qquad \lfloor \alpha^i \rfloor u_{\lfloor \alpha^i \rfloor} \leq n\, u_n \leq \lfloor \alpha^{i+1} \rfloor u_{\lfloor \alpha^{i+1} \rfloor}$$

Dividing by n and using $\lfloor \alpha^i \rfloor \leq n \leq \lfloor \alpha^{i+1} \rfloor$ we get

$$\forall n \geq 1 \qquad \frac{\lfloor \alpha^i \rfloor}{\lfloor \alpha^{i+1} \rfloor} u_{\lfloor \alpha^i \rfloor} \leq u_n \leq \frac{\lfloor \alpha^{i+1} \rfloor}{\lfloor \alpha^i \rfloor} u_{\lfloor \alpha^{i+1} \rfloor} .$$

Therefore, by taking $n \to \infty$ and using ②, we get

$$\frac{\ell}{\alpha} \leq \liminf u_n \leq \limsup u_n \leq \alpha \ell .$$

Since $\alpha > 1$ is arbitrary, this concludes

$$\lim_{n \to \infty} u_n = \ell$$

■

We now give a proof of the LLN when $X_i \in L^2$, $X_i \geq 0$.

For every $\alpha > 1$ $\alpha \in \mathbb{Q}$ and $\varepsilon > 0$, we have

$$\sum_{i \geq 1} P\left(\left| \frac{S_{\lfloor \alpha^i \rfloor}}{\lfloor \alpha^i \rfloor} - m \right| \geq \varepsilon \right) \leq \frac{\mathrm{Var}(X_1)}{\varepsilon^2} \sum_{i \geq 1} \frac{1}{\lfloor \alpha^i \rfloor} < \infty ,$$

Hence, for every $\alpha \in (1, \infty) \cap \mathbb{Q}$, we have

$$\lim_{i \to \infty} \frac{S_{\lfloor \alpha^i \rfloor}}{\lfloor \alpha^i \rfloor} = m \quad \text{a.s.}$$

Since $S_n \nearrow$, by the lemma we get

$$\lim_{n \to \infty} \frac{S_n}{n} = m \quad \text{a.s.} .$$

It remains to prove the result without the assumption $X_1 \geq 0$.

Decompose for every $n \geq 1$

$$X_n = X_n^+ - X_n^- \quad \text{where } X_n^+, X_n^- \geqslant 0.$$

The proof follows by applying the LLN to $(X_n^-)_{n\geq 1}$ and (X_n^+), which holds because the sequences are also in L^2, pairwise independent and identically distributed.

4 L^1 - PROOF.

Assumption : $X_1, X_2, \ldots \in L^1$ pairwise independent, id. distributed

Notation : $X = X_1$

Pb: If $X \in L^1 \setminus L^2$ "$\mathrm{Var}(X)$" $= E(X^2) - E(X)^2 = +\infty$

We cannot say $\mathrm{Var}\left(\frac{S_n}{n}\right) << 1 \ldots$

We will use a truncation argument.

Step 0 : WLOG $X \geqslant 0$ a.s.

(as in the L^2- case)

Step 1: truncation.

Define for every $n \geq 1$

$$Y_n := X_n \mathbb{1}_{X_n \leq n}$$

$$\sum_{n \geq 1} P(X_n \neq Y_n) = \sum_{n \geq 1} \underbrace{P(X_n > n)}_{\leq \int_{n-1}^{n} P(X \geq x)\, dx}$$

$$\leq \int_0^{\infty} P(X \geq x)\, dx = E(X) < \infty.$$

By Borel-Cantelli I, almost surely

$$\exists n_0 \text{ s.t. } \forall n \geq n_0 \quad X_n = Y_n.$$

It suffices to prove

$$\frac{Y_1 + \dots + Y_n}{n} \xrightarrow[n \to \infty]{a.s} m.$$

Step 2: Cesaro

Define for every $n \geq 1$

$$m_n := E(Y_n) = E(X \mathbb{1}_{X \leq n})$$

By dominated convergence $m_n \xrightarrow[n \to \infty]{} m$

By Cesaro Theorem,

$$\frac{m_1 + \cdots + m_n}{n} \xrightarrow[n \to \infty]{} m.$$

Defining $\Sigma_n := Y_1 + \cdots + Y_n$ for every $n \geq 1$,

this implies

$$\frac{E(\Sigma_n)}{n} \xrightarrow[n \to \infty]{} m.$$

Step 3: Chebyshev inequality

Let $\varepsilon > 0$. For every $n \geq 1$, we have

$$P\left(\left|\frac{\Sigma_n - E(\Sigma_n)}{n}\right| \geq \varepsilon\right) \leq \frac{Var(Y_1) + \cdots + Var(Y_n)}{n^2 \varepsilon^2}$$

$$\leq \frac{E(Y_1^2) + \cdots + E(Y_n^2)}{n^2 \varepsilon^2}$$

$Var(Y_i) \leq E(Y_i^2)$

$$\leq \frac{1}{n \varepsilon^2} E\left(Y_1^2 \mathbb{1}_{Y_1 \leq n}\right)$$

$i \leq n \Rightarrow E(Y_i^2) \leq E(Y_n^2)$

Step 4. using subsequences and Ritvik's trick

Let $\alpha > 1$. Let $\varepsilon > 0$.

$$\varepsilon^2 \sum_{i \geq 0} P\left(\left| \frac{\Sigma_{\lfloor \alpha^i \rfloor} - E(\Sigma_{\lfloor \alpha^i \rfloor})}{\lfloor \alpha^i \rfloor} \right| \geq \varepsilon \right)$$

$$\leq \sum_{i \geq 0} \frac{1}{\lfloor \alpha^i \rfloor} E\left(X^2 1_{X \leq \lfloor \alpha^i \rfloor} \right)$$

$$\leq \alpha \times \sum_{i \geq 0} \frac{1}{\alpha^i} E\left(X^2 1_{X \leq \alpha^i} \right)$$

$$\overset{\text{Fubini}}{=} \alpha \, E\left(X_1^2 \times \underbrace{\sum_{i : \alpha^i \geq X} \frac{1}{\alpha^i}}_{\overset{*}{\leq} \frac{\alpha}{\alpha - 1} \times \frac{1}{X} \cdot 1_{X > 0}} \right)$$

$$\leq \frac{\alpha^2}{\alpha - 1} E(X) < \infty$$

Hence,

$$\forall \alpha > 1 \quad \frac{\Sigma_{\lfloor \alpha^i \rfloor}}{\lfloor \alpha^i \rfloor} - \frac{E(\Sigma_{\lfloor \alpha^i \rfloor})}{\lfloor \alpha^i \rfloor} \xrightarrow[i \to \infty]{a.s} 0$$

* see lemma at the end of the section

i.e. by Step 2.

$$\forall \alpha > 1 \qquad \frac{\Sigma_{\lfloor \alpha^i \rfloor}}{\lfloor \alpha^i \rfloor} \xrightarrow[i \to \infty]{a.s} m.$$

Conclusion.

By the geometric subsequence lemma (applied to $\left(\frac{\Sigma_n}{n}\right)$), we conclude

$$\lim_{n \to \infty} \frac{\Sigma_n}{n} = m \qquad a.s. \qquad \blacksquare$$

Lemma (Ritwik's trick)

$$\boxed{\begin{array}{l} \text{Let } x > 0,\ \alpha > 1. \\ \displaystyle\sum_{i : \alpha^i \geq x} \frac{1}{\alpha^i} \leq \frac{\alpha}{\alpha - 1} \cdot \frac{1}{x}. \end{array}}$$

Proof. Let $i_0 \in \mathbb{N}$ s.t. $\alpha^{i_0} \geq x > \alpha^{i_0 - 1}$.

$$\sum_{i : \alpha^i \geq x} \frac{1}{\alpha^i} = \sum_{i \geq i_0} \frac{1}{\alpha^i} = \underbrace{\frac{1}{\alpha^{i_0}}}_{\leq \frac{1}{x}} \cdot \underbrace{\frac{1}{1 - \frac{1}{\alpha}}}_{= \frac{\alpha}{\alpha - 1}}. \qquad \blacksquare$$