CHAPTER 3

L^p - CONVERGENCE

Goal:
- Def of L^p - convergence
- Relation to other notions of convergence.
- Uniform integrability : preparation for martingales

Setup. : . $(\Omega, \mathcal{F}, P)$ fixed probability space.

. $L^p = \{X : \Omega \to \mathbb{R}$ measurable $E(|X^p|) < \infty\}$, $p \geq 1$.

Reminder from measure theory. $p \geq 1$

Let $X \sim Y$ if $X = Y$ a.s.

$(L^p/_{\sim}, \|\cdot\|_p)$ is a complete normed vector space,

with $\|X\|_p = (\int X^p \, dP)^{1/p} = E(X^p)^{1/p}$.

The notion of L^p - convergence for r.v. is the one corresponding to $\|\cdot\|_p$...

Since Ω has $P(\Omega) < \infty$, we have the continuous inclusion $L^q \subset L^p$ if $q \geq p$.

1 DEFINITION.

<u>Def.</u> Let $X, X_1, X_2, \dots \in L^p$, $p \geq 1$. We write

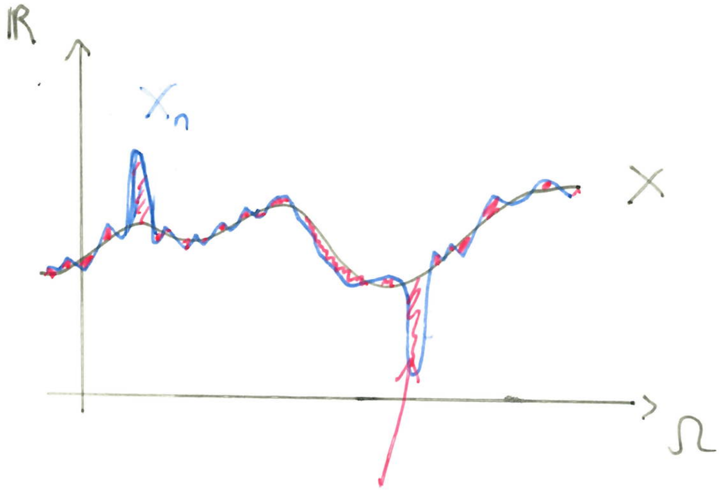
$$X_n \xrightarrow[n \to \infty]{L^p} X$$

if $\lim_{n \to \infty} E(|X_n - X|^p) = 0$.

<u>Rks.</u> • $X_n \xrightarrow[n \to \infty]{L^p} X \iff \|X_n - X\|_p \to 0$

• $\left(X_n \xrightarrow{L^p} X, \ X_n \xrightarrow{L^p} Y\right) \implies X = Y$ a.s.

<u>Interpretation</u> $p = 1$



$|\text{Area}| \to 0$

Prop. Let $1 \leq p \leq q$, $X, X_1, X_2, \dots \in L^q$.

$$\left(X_n \xrightarrow[n\to\infty]{L^q} X\right) \Rightarrow \left(X_n \xrightarrow[n\to\infty]{L^p} X\right)$$

Proof. $E\left(|X_n - X|^p\right) = E\left(|X_n - X|^{q \times \frac{p}{q}}\right)$

$$\overset{\text{Jensen}}{\leq} E\left(|X_n - X|^q\right)^{\frac{p}{q}}$$

$\square$

Rk: $\|X\|_q \leq \|X\|_p$ if $1 \leq p \leq q$ because $P(\Omega) = 1$.

can be proved with Hölder or Jensen.

Examples.

- $X_n \sim \mathrm{Ber}(\alpha_n)$ $\alpha_n \in [0,1]$

$E(|X_n|^p) = \alpha_n$

$X_n \xrightarrow{L^p} 0 \iff \alpha_n \to 0$

- $Y_n = n \times X_n$, $E(|Y_n|^p) = n^p \alpha_n$

$Y_n \xrightarrow{L^p} 0 \iff n^p \alpha_n \to 0$

2. Relation to convergence in probability.

Prop. Let $X, X_1, X_2, \dots \in L^1$,

$$\left(X_n \xrightarrow{L^1} X\right) \Rightarrow \left(X_n \xrightarrow{P} X\right)$$

Proof. $E(|X_n - X| \wedge 1) \leq E(|X_n - X|)$

$$\leq E\left(|X_n - X|^p\right)^{\frac{1}{p}} \quad \blacksquare$$

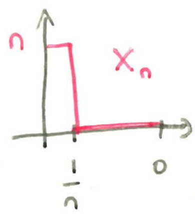
Rk: The reciprocal is not true in general.

Example 1 : $X_n = \begin{cases} 0 & \text{with prob. } 1 - \frac{1}{n} \\ n & \text{———} \quad \frac{1}{n} \end{cases}$

$$X_n \xrightarrow[n \to \infty]{P} 0 \quad \text{but} \quad E(|Y_n|) = n \times \frac{1}{n} = 1 \not\to 0.$$

Example 2 : $\Omega = [0, 1] \quad P = \mathcal{L}eb.$

$$X_n(\omega) = n \times 1_{\omega \leq \frac{1}{n}}$$



$$X_n \xrightarrow[n \to \infty]{a.s} 0 \quad \text{but} \quad E(|X_n|) = 1 \underset{n \to \infty}{\not\to} 0$$

3 RELATION TO A.S. CONVERGENCE

Convergence in L^p does not imply a.s convergence in general.

Example 1

$(X_n)_{n\geq 1}$ independent.

$X_n \sim Ber\left(\frac{1}{n}\right)$

$X_n \xrightarrow[n\to\infty]{L^p} 0$ but $X_n \not\xrightarrow[n\to\infty]{a.s} 0$.

Convergence a.s. does not imply convergence in L^p in general. See Example 2 in previous section, or the following example.

Example 2 : $X_n = \begin{cases} 0 & \text{with proba } 1-\frac{1}{2^n} \\ 2^n & \text{———} \quad \frac{1}{2^n} \end{cases}$

$X_n \xrightarrow{a.s.} 0 \quad \left(\sum P(X_n > \varepsilon) < \infty \ \forall \varepsilon \right)$

but $E(|X_n|) = 1 \not\to 0$.

Rk: L^p - cv $\Rightarrow$ cv in P. $\Rightarrow$ a.s cv along subsequence.

4 UNIFORM INTEGRABILITY.

1 Let $X, X_1, X_2, \ldots$ be such that $\lim_{n\to\infty} X_n = X$ a.s.

The dominated convergence theorem guarantes that the convergence also holds in L^1 if

$$\exists Z \in L^1 .\ \forall n \quad |X_n| \leq Z \text{ a.s.}$$

(because this implies $|X| \leq Z$ a.s and therefore $|X_n - X| \leq 2Z$ a.s.)

Questions: - Can we find an alternative weaker condition than domination?

- Does it remain true if we replace the a.s. cv by cv in probability?

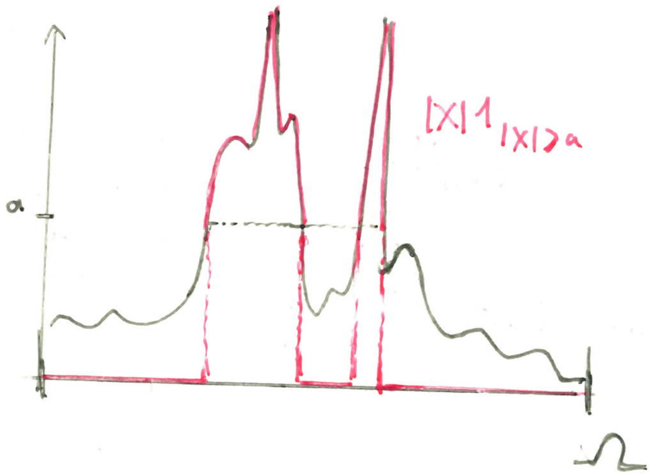
→ To answer these question, we introduce the notion of uniform integrability.

Rk: Let X be a random variable. We have

$$X \text{ "integrable"} \iff E(|X|) < \infty$$

$$\iff \lim_{a \to \infty} E(|X| 1_{|X|>a}) = 0$$

($\Rightarrow$ by cv dominated, $\Leftarrow$ using $|X| \leq a + |X| 1_{|X|>a}$)



Rk: $$E(|X| 1_{|X| \geq a}) = \int_0^\infty P(|X| 1_{|X| \geq a} \geq x)\, dx$$

$$= \int_a^\infty P(|X| \geq x)\, dx \underset{a \to \infty}{\longrightarrow} 0$$

"tail integral"

Example: $X \sim Exp(\lambda)$

$$E(X 1_{X \geq a}) = \int_a^\infty e^{-\lambda x} dx = \frac{1}{\lambda} e^{-\lambda a}.$$

The "more integrable" X is, the faster the decay of $E(|X| 1_{|X| \geq a})$ is.

<u>Def.</u> A family $\{X_i\}_{i\in I}$ of random variables is uniformly integrable (UI) if

$$\sup_{i\in I} E\left(|X_i| 1_{|X_i|\geq a}\right) \xrightarrow[a\to\infty]{} 0$$

<u>Rk</u>: $\Big((X_i)_{i\in I}$ is UI$\Big)$

$$\Longleftrightarrow \left(\forall \varepsilon>0 \quad \exists a>0 \text{ s.t. } \forall i\in I \quad E\left(|X_i| 1_{|X_i|\geq a}\right) \leq \varepsilon \right)$$

<u>Examples</u>:

- If $X \in L^1$ $\{X\}$ is uniformly integrable.
- If $X_1, \dots, X_N \in L^1$ $\{X_1, \dots, X_N\}$ is UI.
- $X_\lambda \sim \mathrm{Exp}(\lambda)$

  $\{X_\lambda\}_{\lambda\geq 1}$ is U.I.

  $\{X_\lambda\}_{\lambda>0}$ is not UI

  $$\left(\sup_{\lambda>0} \frac{1}{\lambda} e^{-\lambda a}\right) = +\infty$$

5 SUFFICIENT CONDITIONS FOR UI

PROPOSITION. Let $(X_i)_{i\in I}$ be rvs. Suppose $\exists$ a rv $Z \in L^1$ s.t.

$$\forall i \in I \quad |X_i| \leq Z. \qquad (*)$$

Then $(X_i)_{i\in I}$ is UI

Proof. By $(*)$ we have,

$$\forall a>0 \ \forall i\in I \ E\left(|X_i| 1_{|X_i|\geq a}\right) \leq E\left(|Z| 1_{|Z|\geq a}\right).$$

So, $\sup\limits_{i\in I} E\left(|X_i| 1_{|X_i|\geq a}\right) \leq E\left(|Z| 1_{|Z|\geq a}\right) \xrightarrow[a\to\infty]{} 0.$ $\square$

(because $Z\in L^1$)

PROPOSITION. Let $p>1$ be s.t.

$$\sup_{i\in I} E\left(|X_i|^p\right) < \infty.$$

Then $(X_i)_{i\in I}$ is UI.

Proof. $\sup\limits_{i\in I} E\left(|X_i| 1_{|X_i|\leq a}\right)$

$$\leq \sup_{i\in I} E\left(\frac{|X_i|^{p-1}}{a^{p-1}} |X_i| 1_{|X_i|\leq a}\right)$$

$$\leq \frac{1}{a^{p-1}} \sup_{i\in I} E\left(|X_i|^p\right) \xrightarrow[a\to\infty]{} 0$$

$\square$

6 FROM CONVERGENCE IN PROBABILITY TO CONVERGENCE IN L^1

MAIN THEOREM. Let $(X_n)_{n\geq 1}$, X be rvs s.t. $X_n \xrightarrow[n\to\infty]{P} X$.
Then the following are equivalent.

(i) $(X_n)_{n\geq 1}, X \in L^1$ and $X_n \xrightarrow[n\to\infty]{L^1} X$.

(ii) $(X_n)_{n\geq 1}$ is UI

COROLLARY. Let $(X_n)_{n\geq 1}$, X be rvs s.t. $(X_n)_{n\geq 1}$ is UI, and $X_n \xrightarrow[n\to\infty]{a.s.} X$. Then $E(X_n) \to E(X)$.

Proof. The Main Theorem implies that $X_n \xrightarrow{L^1} X$. So,

$$|E(X) - E(X_n)| = |E(X - X_n)| \leq E(|X - X_n|) \xrightarrow[n\to\infty]{} 0. \quad \square$$

APPLICATION (L^1 convergence in LLN)

THEOREM. Let $(X_n)_{n\geq 1}$ be identical, pairwise ind., L^1 rvs. Then

$$\frac{X_1 + \cdots + X_n}{n} \xrightarrow[n\to\infty]{L^1} E(X_1).$$

Proof. By SLLN and the Main Theorem, it is enough to show that $\left(\frac{S_n}{n}\right)_{n\geq 1}$ is UI, where $S_n = X_1 + \cdots + X_n$.

Fix $\varepsilon > 0$. Let $a > 0$. Then we have the following for all $n \geq 1$.

$$E\left(\left|\frac{S_n}{n}\right| 1_{\left|\frac{S_n}{n}\right| \geq a}\right) \leq E\left(\frac{|X_1| + \cdots + |X_n|}{n} 1_{\left|\frac{S_n}{n}\right| \geq a}\right)$$

$$= \frac{1}{n} \sum_{i=1}^{n} E\left(|X_i| 1_{\left|\frac{S_n}{n}\right| \geq a}\right). \qquad (1)$$

We estimate each term as follows.

$$E\left(|X_i| 1_{\left|\frac{S_n}{n}\right| \geq a}\right)$$

$$= E\left(|X_i| 1_{\left|\frac{S_n}{n}\right| \geq a} 1_{|X_i| \geq \sqrt{a}}\right) + \underbrace{E\left(|X_i| 1_{\left|\frac{S_n}{n}\right| \geq a} 1_{|X_i| \leq \sqrt{a}}\right)}$$

$$\leq \sqrt{a}\, E\left(1_{|S_n - X_i| \geq (n-1)a}\right)$$

$$\leq \sqrt{a}\, P\left(|X_1| + \cdots + |X_n| - |X_i| \geq (n-1)a\right)$$

$$\overset{\text{(Markov)}}{\leq} \sqrt{a}\, \frac{E(|X_1|)}{a} = \frac{E(|X_1|)}{\sqrt{a}}.$$

So choosing $a > 0$ large enough s.t. $E(|X_1| 1_{|X_1| \geq \sqrt{a}}) \leq \frac{\varepsilon}{2}$ and $E(|X_1|)/\sqrt{a} \leq \frac{\varepsilon}{2}$, and using (1) shows

$\forall n \geq 1$ $E\left(\left|\frac{S_n}{n}\right| 1_{\left|\frac{S_n}{n}\right| \geq a}\right) \leq \varepsilon$. $\square$

Proof of MAIN THEOREM.

((i) $\Leftarrow$ (ii)) Step 1: We first prove that $\forall a > 0$

$$E(|X_n - X| \wedge a) \xrightarrow[n\to\infty]{} 0 .$$

Fix $a > 0$. Since $\frac{X_n}{a} \xrightarrow{P} \frac{X}{a}$, we have

$$E(|X_n - X| \wedge a) = a\, E\left(\left| \frac{X_n}{a} - \frac{X}{a} \right| \wedge 1 \right) \xrightarrow[n\to\infty]{} 0 .$$

Step 2: Let $\varepsilon > 0$. Since $(X_n)_{n\geq 1}$ is UI, we can fix $a > 0$ s.t.

$$\forall n \geq 1 \quad E\left(|X_n| 1_{|X_n| \geq a} \right) \leq \varepsilon .$$

2.1 Choosing a subsequence $n(k)$ s.t. $X_{n(k)} \xrightarrow[n\to\infty]{a.s.} X$,

we get $\varepsilon \geq \liminf_{n\to\infty} E\left(|X_n| 1_{|X_n|\geq a} \right) \geq E\left(|X| 1_{|X| \geq a} \right)$

Fatou's Lemma.

2.2 $E(|X_n - X|) =$

$$\underbrace{E\left(|X_n - X| 1_{\max(|X_n|, |X|) \leq a} \right)}_{\leq E(|X_n - X| \wedge 2a)} + \underbrace{E\left(|X_n - X| 1_{\max(|X_n|,|X|) > a} \right)}$$

$\leq E(|X_n - X| \wedge 2a) \xrightarrow[n\to\infty]{} 0$ (by step 1)

$$\leq 2E\left(|X_n| 1_{|X_n| \geq a} \right) + 2E\left(|X| 1_{|X| \geq a} \right) \leq 4\varepsilon .$$

Hence, $\limsup_{n\to\infty} E(|X_n - X|) \leq 4\varepsilon$.

$\big((i) \Rightarrow (ii)\big)$ Fix $\varepsilon > 0$.

STEP 1 Let $a > 0$ be s.t.

I $E\left(|X| \mathbb{1}_{|X| \geq a}\right) \leq \varepsilon$,

II & $\forall n \geq 1$ $E\left(|X_n - X| \mathbb{1}_{|X_n - X| \geq a}\right) \leq \varepsilon$.

We can find $a > 0$ satisfying I because $X \in L^1$.

We can find $a > 0$ satisfying II because $|X_n - X| \xrightarrow[n \to \infty]{L^1} 0$.

Indeed, let $n_0 \geq 1$ be s.t. $\forall n \geq n_0$ $E\left(|X_n - X|\right) \leq \varepsilon$

and let $a > 0$ be s.t. $\sup\limits_{1 \leq n \leq n_0} E\left(|X_n - X| \mathbb{1}_{|X_n - X| > a}\right) \leq \varepsilon$.

STEP 2 : Let $n \geq 1$. Then,

$X_n = X + X_n - X$.

$$E\left(|X_n| \mathbb{1}_{|X_n| \geq 2a}\right) \leq E\left(\left(|X| + |X_n - X|\right) \mathbb{1}_{|X_n| \geq 2a}\right)$$

$$= E\left(\left(|X| + |X_n - X|\right) \mathbb{1}_{|X_n| \geq 2a} \mathbb{1}_{|X| \geq |X_n - X|}\right)$$

$$+$$

$$E\left(\left(|X| + |X_n - X|\right) \mathbb{1}_{|X_n| \geq 2a} \mathbb{1}_{|X_n - X| > |X|}\right)$$

$$\leq 2\,E\left(|X| \mathbb{1}_{|X| \geq a}\right) + 2\,E\left(|X_n - X| \mathbb{1}_{|X_n - X| \geq a}\right)$$

$$\leq 4\varepsilon.$$

□