FOURIER TRANSFORM IN $\mathbb{R}^d$

Ref. Lieb-Loss / Villani / Körner.

Setup. $\mathbb{R}^d$ $\quad \langle x, y \rangle = \sum_{k=1}^{d} x_k y_k$.

Def. $f \in L^1(\mathbb{R}^d)$. For every $t \in \mathbb{R}^d$, we define

$$\hat{f}(t) = \int_{\mathbb{R}^d} f(x)\, e^{i\langle x, t\rangle}\, dx$$

⚠ Proba convention: $e^{-i\langle x,t\rangle}$ or $e^{-2\pi i\langle x,t\rangle}$

are more standard in analysis.

1 CONVOLUTION FORMULA.

$(f * g)(x) = \int_{\mathbb{R}^d} f(x-y)\, g(y)\, dy.$

Prop. Let $f, g \in L^1(\mathbb{R}^d)$. Then $f*g \in L^1(\mathbb{R}^d)$ and

$$\widehat{f * g} = \hat{f} \times \hat{g}.$$

Proof. $\int (f*g)(x)\, e^{i\langle t,x\rangle} dx = \int \left(\int f(x-y)\, e^{i\langle t, x-y\rangle} g(y) e^{i\langle t,y\rangle} dy \right) dx$

$$\overset{\text{Fubini}}{=} \int g(y) e^{ity} \underbrace{\left(\int f(x-y)\, e^{i\langle t, x-y\rangle} dx \right)}_{\overset{z=x-y}{=} \hat{f}(t)} dy$$

$$= \hat{g}(t)\, \hat{f}(t)$$

$\blacksquare$

2 DIFFERENTIAL FORMULA.

Prop Let $f \in L^1(\mathbb{R}^d)$ and assume $\frac{\partial}{\partial x_1} f$ well def. everywhere and $\frac{\partial f}{\partial x_1} \in L^1$, then

$$\forall t \quad \widehat{\frac{\partial}{\partial x_1} f}(t) = -it_1 \hat{f}(t)$$

Proof. $(d=1)$

- $f(x) \xrightarrow[|x|\to\infty]{} 0$. Indeed $f(x) = f(0) + \int_0^x f'(y)\,dy$

where $f' \in L^1$. Hence $\lim_{x\to\infty} f(x)$ exists.

Since $f \in L^1$ we must have $\lim_{x\to\infty} f(x) = 0$.

- By integration by part, we have

$$\underbrace{\int_{-a}^{a} f'(x) e^{itx}\,dx}_{\downarrow \; \hat{f'}(t)} = \underbrace{\left[f(x) e^{itx} \right]_{-a}^{a}}_{\downarrow_{a\to\infty} \; 0} - (it) \underbrace{\int_{-a}^{a} f(x) e^{itx}\,dx}_{\downarrow \; \hat{f}(t)}$$

3 TRANSPOSITION FORMULA.

Prop. Let $f, \varphi \in L^1(\mathbb{R}^d)$. Then $\hat{f}\varphi$ and $f\hat{\varphi}$ are in L^1

and

$$\int_{\mathbb{R}^d} \hat{f}\varphi = \int_{\mathbb{R}^d} f\,\hat{\varphi}.$$

Proof.
$$\int_{\mathbb{R}^d} |\hat{f}||\varphi| \leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |f(x)e^{i\langle t,x\rangle}|dx\ |\varphi(t)|dt$$

$$= \int |f(x)|dx \times \int |\varphi(t)|\,dt. < \infty$$

By Fubini Thm

$$\int_{\mathbb{R}^d} \hat{f}(t)\,\varphi(t)\,dt = \iint_{\mathbb{R}^d\times\mathbb{R}^d} f(x)\,\varphi(t)\,e^{i\langle t,x\rangle}dt\,dx$$

$$= \int_{\mathbb{R}^d} f(x)\,\hat{\varphi}(x)\,dx. \quad \blacksquare$$

4 FOURIER TRANSFORM OF GAUSSIAN DENSITY.

<u>Prop.</u> Let $g(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$.

$$\forall t \in \mathbb{R} \quad \hat{g}(t) = e^{-\frac{t^2}{2}}$$

<u>Proof.</u> $\hat{g}(t) = \int \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} e^{itx} dx = e^{-\frac{t^2}{2}} \underbrace{\frac{1}{\sqrt{2\pi}} \int e^{-\frac{1}{2}(x-it)^2} dx}_{=: \,\Psi(t)}$.

$\Psi(0) = 1$. By derivation under $\int$, we have

$$\forall t \in \mathbb{R} \quad \Psi'(t) = \frac{i}{\sqrt{2\pi}} \int (x-it)\, e^{-\frac{1}{2}(x-it)^2} dx$$

$$= -\frac{i}{\sqrt{2\pi}} \left[e^{-\frac{1}{2}(x-it)^2} \right]_{x=-\infty}^{x=+\infty} = 0$$

Hence $\forall t \in \mathbb{R} \quad \Psi(t) = 1$. ∎

5 INVERSION FORMULA

Thm. Let $f : \mathbb{R}^d \to \mathbb{C}$ continuous bounded.
If $f \in L^1(\mathbb{R}^d)$ and $\hat{f} \in L^1(\mathbb{R}^d)$, then

$$\forall x \in \mathbb{R}^d \quad \hat{\hat{f}}(x) = f(-x)$$

Proof. Let $Z = (Z_1, \dots, Z_d)$ where $Z_1, \dots, Z_d$ iid $\mathcal{N}(0,1)$. Let $x \in \mathbb{R}^d$

For every $n \geq 1$

$$E\left(\hat{\hat{f}}\left(x + \frac{1}{n} Z\right)\right) = \int_{\mathbb{R}^d} \hat{\hat{f}}\left(x + \frac{1}{n} z\right) g(z_1) \dots g(z_d)\, dz$$

$$\overset{(*)}{=} (2\pi)^d \int_{\mathbb{R}^d} f\left(-x + \frac{1}{n} z\right) g(z_1) \dots g(z_d)\, dz.$$

transposition formula

$$= (2\pi)^d E\left(f\left(-x + \frac{1}{n} Z\right)\right)$$

By dominated convergence (f and $\hat{f}$ are both continuous bounded), we can take the limit as $n \to \infty$ to get $\hat{\hat{f}}(x) = f(-x)$ ∎

(*) Here we use $\hat{\hat{G}}(z) = (2\pi)^d G(z)$ if $G(z) = g(z_1) \dots g(z_d)$

and $\hat{\hat{f}}\left(x + \frac{1}{n} z\right) = \hat{\hat{f}}_{x,n}(z)$ where $f_{x,n}(z) = f\left(-x + \frac{1}{n} z\right)$.

Rk: If $f \in \mathcal{S}$ (Schwartz space) or $f \in \mathcal{C}_c^\infty$ (infinitely differentiable with compact support), then f satisfies the hypotheses of the theorem.

CHAPTER 5.

CHARACTERISTIC FUNCTIONS.

Goals:
- link with standard Fourier analysis.
- dictionary "property of X" $\leftrightarrow$ "property of φ_X"
- applications : characterisation of laws / computation of moment
- distribution viewpoint : use of test functions.

Setup. $(\Omega, \mathcal{F}, P)$ fixed probability space.

- $\mathbb{R}^d$, $d \geq 1$ equiped with $\mathcal{B}(\mathbb{R}^d)$, $\langle x ; y \rangle = \sum_{i=1}^{d} x_i y_i$.
- $\mathcal{C}_c^\infty = \{ f : \mathbb{R}^d \to \mathbb{R} \ \mathcal{C}^\infty, \text{ compact support} \}$

↳ if $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$ multi index. $f \in \mathcal{C}_c^\infty$

$$D^\alpha f(x) := \frac{\partial}{\partial^{\alpha_1} x_1} \cdots \frac{\partial}{\partial^{\alpha_d} x_d} f$$

"differential operator"

Expection for r.v. with complex values, if $Z : \Omega \to \mathbb{C}$ $\quad Z = X + iY$

We have $Z \in L^1$ if $E(|Z|) < \infty$ (i.e. $X, Y \in L^1$)

In this case we define $E(Z) = E(X) + i E(Y)$.

1) DEFINITION.

Def. Let X be a r.v. with values in $\mathbb{R}^d$. The characteristic function of X is the function $\varphi_X : \mathbb{R}^d \longrightarrow \mathbb{C}$, defined by

$$\forall t \in \mathbb{R}^d \qquad \boxed{\varphi_X(t) = E\left(e^{i<t,X>}\right)}$$

Rk: $z = e^{i<t,X>}$ satisfies $|z| = 1 \in L^1$., hence $\varphi_X(t)$ is always well defined, and

$$\forall t \qquad |\varphi_X(t)| \leq E\left(\left|e^{i<t,X>}\right|\right) \leq 1 .$$

Rk: φ_X is a property of the measure $\mu = \mu_X$:

$$\varphi_X(t) = \int e^{i<t,x>} \, d\mu_X(x)$$

$\hookrightarrow$ $\varphi_X = \hat{\mu}_X$ in the sense of distribution.

<u>Elementary properties</u> Let X be a r.v. with values in $\mathbb{R}^d$

(i) φ is uniformly continuous in $\mathbb{R}^d$.

(ii) $\varphi(0) = 1$

(iii) $\forall t \in \mathbb{R}^d \quad \varphi_X(-t) = \overline{\varphi_X(t)}$

(in particular $X \overset{(law)}{=} -X \Rightarrow \varphi_X$ is real and even)

<u>Proof</u>. (i) Let $h \in \mathbb{R}^d$

$$\forall t \in \mathbb{R}^d \quad |\varphi_X(t+h) - \varphi_X(t)| = |E(e^{i<t+h,X>} - e^{i<t,X>})|$$

$$= |E(e^{i<t,X>}(e^{i<h,X>} - 1))|$$

$$\leq \underbrace{E(|e^{i<h,X>} - 1|)}$$

Dom. Cv. $\downarrow h \to 0$

0

(ii) $\varphi(0) = E(1) = 1$.

(iii) $\varphi_X(-t) = E(e^{i<-t,X>}) = \overline{E(e^{i<t,X>})}$.

Rk: If X is a real random variable with density f, then

$$\forall t \in \mathbb{R} \qquad \varphi_X(t) = \int_{\mathbb{R}} e^{ixt} f(x)\, dx = \hat{f}(t).$$

Fourier transform of $f \in L^1$
(with probability convention)

Example 1 $X \sim \mathcal{N}(0,1)$ $\left(\text{density } g(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\right)$

$$\varphi_X(t) = \hat{g}(t) = e^{-t^2/2}$$

Example 2 $Z \in \mathbb{Z}^d$ uniform in $\{\pm e_k\}_{k=1}^d$

$d = 2$  $P(Z = (0,1)) = \frac{1}{4}$

$$\varphi_Z(t) = \sum_{k=1}^{d} \frac{1}{2d} e^{i\langle t, e_k\rangle} + \frac{1}{2d} e^{i\langle t, -e_k\rangle} \qquad t = (t_1, \dots, t_d)$$

$$\varphi_Z(t) = \frac{1}{d} \sum_{k=1}^{d} \cos(t_k)$$

Prop. $\boxed{\text{Let } \alpha \in \mathbb{R},\ u \in \mathbb{R}^d,\ X \text{ r.v. in } \mathbb{R}^d. \quad \forall t \in \mathbb{R}^d \ \boxed{\varphi_{\alpha X + u}(t) = e^{i\langle u, t\rangle} \times \varphi_X(\alpha t)}}$

Proof $\mathbb{E}\left(e^{i\langle \alpha X + u, t\rangle}\right) = \mathbb{E}\left(e^{i\langle u, t\rangle} \times e^{i\langle \alpha X, t\rangle}\right)$

$$= e^{i\langle u, t\rangle} \times \mathbb{E}\left(e^{i\langle X, \alpha t\rangle}\right). \blacksquare$$

Application. Let $m \in \mathbb{R}$, $\sigma > 0$. $X \sim \mathcal{N}(m, \sigma^2)$.

$X \overset{(law)}{=} m + \sigma Y$ where $Y \sim \mathcal{N}(0,1)$.

$$\boxed{\varphi_X(t) = e^{imt}\, e^{-\frac{\sigma^2}{2}t^2}}$$

2 CONVOLUTION FORMULA

Rk: X density f, Y density g independent real r.v..

$X+Y$ has density $f * g$. Hence

$$\varphi_{X+Y} = \widehat{f * g} = \hat{f} \times \hat{g} = \varphi_X \times \varphi_Y.$$

Does this generalize to arbitrary r.v.s ?

Prop. Let X, Y be two indep. r.v.s in $\mathbb{R}^d$.

$$\forall t \in \mathbb{R}^d \quad \varphi_{X+Y}(t) = \varphi_X(t)\,\varphi_Y(t)$$

Proof. $\forall t \in \mathbb{R}^d$ $E\left(e^{i\langle t, X+Y\rangle}\right) = E\left(e^{i\langle t, X\rangle} \times e^{i\langle t, Y\rangle}\right)$

$$\overset{\text{(indep.)}}{=} E\left(e^{i\langle t, X\rangle}\right) E\left(e^{i\langle t, Y\rangle}\right) \blacksquare$$

Application:

Let $X \sim \mathcal{N}(m_1, \sigma_1^2)$ $Y \sim \mathcal{N}(m_2, \sigma_2^2)$ indep. $m_i \in \mathbb{R}$, $\sigma_i > 0$

$$\forall t \in \mathbb{R} \quad \varphi_{X+Y}(t) = e^{i(m_1+m_2)t}\, e^{-(\sigma_1^2+\sigma_2^2)\frac{t^2}{2}}$$

Prop. Let $X_1, \dots, X_n$ indep r.v. in $\mathbb{R}^d$.

$$\forall t \in \mathbb{R}^d \quad \varphi_{X_1+\dots+X_n}(t) = \varphi_{X_1}(t) \times \dots \times \varphi_{X_n}(t).$$

Application $S_n = X_1 + \dots + X_n$ where (X_i) iid unif. in $\{\pm e_k\}_{k=1}^d$.
(SRW in $\mathbb{Z}^d$)

$$\varphi_{S_n}(t) = \varphi(t)^n \quad \text{where } \varphi(t) = \frac{1}{d}\sum_{k=1}^{d} \cos(t_k).$$

3 φ IS CHARACTERISTIC (OF THE LAW)

Th. Let X, Y be two r.v.s in $\mathbb{R}^d$.

$$\boxed{\varphi_X = \varphi_Y \iff P_X = P_Y.}$$

Application $X \sim \mathcal{N}(m_1, \sigma_1^2)$, $Y \sim \mathcal{N}(m_2, \sigma_2^2)$ indep.

$$X+Y \sim \mathcal{N}(m_1+m_2, \sigma_1^2+\sigma_2^2)$$

(because $\varphi_{X+Y} = \varphi_Z$ where $Z \sim \mathcal{N}(m_1+m_2, \sigma_1^2+\sigma_2^2)$)

Other applications $\quad \mathrm{Poi}(\alpha) * \mathrm{Poi}(\beta) = \mathrm{Poi}(\alpha+\beta)$ (see exercises).

$$\Gamma(\lambda, m) * \Gamma(\lambda, n) = \Gamma(\lambda, m+n)$$

Lemma 1 (Parseval-type formula).

> Let f continuous bounded $f, \hat{f} \in L^1(\mathbb{R}^d)$.
> X r.v. with values in $\mathbb{R}^d$.
>
> $$E(f(X)) = \int_{\mathbb{R}^d} \hat{f}(t)\, \overline{\varphi_X(t)}\, dt.$$

Rk: In the distribution sense

$$\text{“} (f, \varphi) = (\hat{f}, \hat{\varphi}) \text{”}.$$

Proof: By inversion Theorem,

$$E(f(X)) = E(\hat{\hat{f}}(-X))$$

$$= E\left(\int_{\mathbb{R}^d} \hat{f}(t)\, e^{-i\langle t, X\rangle}\, dt\right)$$

$$\overset{\text{Fub.}}{=} \int_{\mathbb{R}^d} \hat{f}(t)\, \underbrace{E\left(e^{-i\langle t, X\rangle}\right)}_{=\overline{\varphi_X(t)}}\, dt \qquad \blacksquare$$

Lemma 2 Let X, Y be two r.v.s in $\mathbb{R}^d$.

If $\forall f \in \mathcal{C}_c^\infty \quad E(f(X)) = E(f(Y))$,

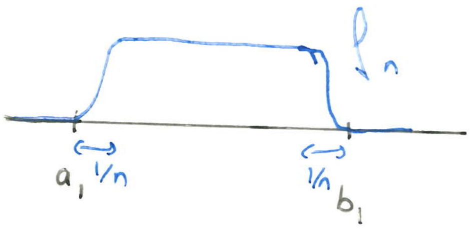
then $\mu_X = \mu_Y$.

Proof: Let $R = (a_1, b_1) \times \dots \times (a_d, b_d) \quad a_i < b_i$.

Let $f_n \in \mathcal{C}_c^\infty, \ n \geq 1$ s.t.

$$f_n \underset{n\to\infty}{\uparrow} 1_R$$

Illustration in $d = 1$



By hypotheses.

$$\forall n \quad E(f_n(X)) = E(f_n(Y))$$

Hence, by monotone convergence,

$$\underbrace{P(X \in R)}_{\mu_X(R)} = \underbrace{P(Y \in R)}_{\mu_Y(R)}.$$

This holds for arbitrary R in the π-system

$$\{(a_1, b_1) \times \dots \times (a_d, b_d),\ a_i < b_i\},$$

which generates $\mathcal{B}(\mathbb{R}^d)$. Therefore $\mu_X = \mu_Y$. ∎

The Thm follows from Lemma 1 and Lemma 2.

4 DISCRETE INVERSION FORMULA.

Thm. Assume X r.v. s.t $X \in \mathbb{Z}^d$ a.s. (X discrete).

$$\forall x \in \mathbb{Z}^d \quad P(X = x) = (2\pi)^d \int_{(-\pi,\pi]^d} e^{-i\langle t, x\rangle} \varphi_X(t)\, dt.$$

Rk: $d = 1 \qquad X \in \mathbb{Z} \rightsquigarrow p_k = P(X = k)$

$$\varphi_X(t) = \sum p_k e^{ikt} \qquad 2\pi\text{-periodic}$$

$$\hookrightarrow p_k = \text{Fourier coefficient} = \frac{1}{2\pi} \int e^{-itk} \varphi_X(t)\, dt.$$

Pf: $\displaystyle\int_{(-\pi,\pi]^d} e^{-i\langle t,k\rangle}\,\varphi_X(t)\,dt$

$$= \int_{(-\pi,\pi]^d} e^{-i\langle t,k\rangle}\, E\left(e^{i\langle t,X\rangle}\right) dt$$

$$\overset{\text{Fub.}}{=} E\Big(\underbrace{\int_{(-\pi,\pi]^d} e^{i\langle t, X-k\rangle}\,dt}_{(2\pi)^d\, 1_{X=k}}\Big) \qquad \blacksquare$$

Application SRW return probability.

$X_1,\dots,X_n$ iid $\mathcal{U}(\{-1,1\})$ $\quad S_n = X_1+\dots+X_n$. For $n\geq 1$,

$$P(S_{2n}=0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} (\cos t)^{2n}\,dt$$

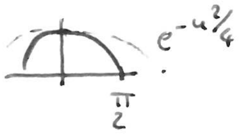
$$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} (\cos t)^{2n}\,dt \qquad \left(\begin{array}{l}\cos(-t)=\cos(t)\\ \cos^2(\pi-t)=\cos^2(t)\end{array}\right)$$

$$= \frac{2}{\pi\sqrt{2n}}\underbrace{\int_0^{\frac{\pi}{2}\times\sqrt{2n}} \left(\cos\left(\frac{u}{\sqrt{2n}}\right)\right)^{2n} du}_{=\int_0^\infty \Psi_n(u)\,du}. \qquad \left(t=\frac{u}{\sqrt{2n}}\right)$$

where $\Psi_n(u) = \cos\left(\frac{u}{\sqrt{2n}}\right)^{2n} 1_{u\leq\sqrt{2n}\times\frac{\pi}{2}}$

$$\forall u \in \mathbb{R}_+ \quad \cos^2\left(\frac{u}{\sqrt{2n}}\right)^{2n} = \left(1 - \frac{u^2}{4n} + o\left(\frac{1}{n}\right)\right)^{2n} \xrightarrow[n\to\infty]{} e^{-\frac{u^2}{2}}$$

$$\forall t \leq \frac{\pi}{2} \quad \cos^2(t) \leq e^{-\frac{u^2}{4}}$$



Hence $\Psi_n(u) \xrightarrow[n\to\infty]{} e^{-\frac{u^2}{2}}$

$$\forall n \quad \Psi_n(u) \leq e^{-\frac{u^2}{2}}$$

By dominated convergence $\displaystyle\int_0^\infty \Psi_n(u)\,du \to \int_0^\infty e^{-u^2/2} = \frac{1}{2}\sqrt{2\pi}$

<u>ccl</u> : $P(S_{2n}=0) \underset{n\to\infty}{\sim} \dfrac{1}{\sqrt{\pi n}}$ ■

<u>Rk 1.</u>

- In Lemma 1, we cannot take $\Psi = 1_{[a,b]}$ in general ($\hat{\Psi} \notin L^1$). To compute $P(X \in [a,b])$, there exists some general formulas using approximation (see Durrett for example, or the exercises)

5 DIFFERENTIAL FORMULA.

Existence of moments of X $\longleftrightarrow$ regularity of φ_X.

Thm: Let X be a real r.v., $k \in \mathbb{N}$.

If $E(|X|^k) < \infty$, then φ is $\mathcal{C}^k$ and

$$\boxed{\forall t \in \mathbb{R} \quad \varphi^{(k)}(t) = E\left((iX)^k e^{itX}\right)}$$

Proof: Derivation under $\int$:

Let $\Psi(t,\omega) = e^{it.X(\omega)}$ $\quad \left(\varphi(t) = \int_\Omega \Psi(t,\omega)\, dP.\right)$

Ψ is k-time continuously differentiable in t and we have the domination for every $\ell \leq k$

$$\forall t, \omega \quad \left|\frac{\partial^\ell \Psi}{\partial t^\ell}(t,\omega)\right| \leq |X(\omega)|^\ell.$$

Hence $\varphi : t \mapsto \int_\Omega \Psi(t,\omega)\, dP$ is $\mathcal{C}^k$ and

$$\varphi^{(k)}(t) = \int \frac{\partial^k \Psi}{\partial t^k}(t,\omega)\, dP$$

$$= \int_\Omega (iX(\omega))^k e^{itX(\omega)}\, dP$$

$$= E\left((iX)^k e^{itX}\right) \qquad \blacksquare$$

Application : Let $X \sim \mathcal{N}(0,1)$.

$$\forall k \in \mathbb{N} \quad E(X^{2k}) = \frac{(2k)!}{2^k k!} = 1 \times 3 \times 5 \times \dots \times (2k-1)$$

Rk: By sym. $E(X^{2k+1}) = 0$.

Proof. Since $E(X^{2k}) < \infty$ φ is $\mathcal{C}^{2k}$, and

by Taylor expansion

$$\varphi(t) = \varphi(0) + \dots + \frac{\varphi^{(2k)}(0)}{(2k)!} \times t^{2k} + o(t^{2k})$$

On the other hand

$$\varphi(t) = e^{-\frac{t^2}{2}} = 1 - \frac{t^2}{2} + \dots + \frac{(-1)^k}{2^k k!} t^{2k} + o(t^{2k})$$

By identifying the coefficient k in the Taylor expansion we get

$$E(X^{2k}) = i^{2k} \varphi^{(2k)}(0) = \frac{(2k)!}{2^k k!} \quad \blacksquare$$

Corollary. Let X be a real r.v. s.t. $E(X) = 0$, $E(X^2) < \infty$.
Then $\varphi = \varphi_X$ is $\mathcal{C}^2$ and, writing $\sigma^2 = \mathrm{Var}(X)$.

$$\varphi(t) = 1 - \frac{\sigma^2 t^2}{2} + o(t^2)$$

Proof: Taylor expansion.

Application (exercise)

Let $X_1, X_2, \dots$ iid with $X_1 \overset{law}{=} -X_1$ $P(X_1 = 1) > 0$

$X_1 \in \mathbb{Z}$ a.s. $E(X_1) = 0$ $Var(X_1) = \sigma^2$

$S_n = X_1 + \dots + X_n$. Show that there exists $c > 0$

$$P(S_{2n} = 0) \underset{n \to \infty}{\sim} \frac{c}{\sqrt{n}}.$$

6 INDEPENDENCE.

Prop. Let $X_1, \dots, X_n$ real random variables. $n \geq 1$.

$(X = (X_1, \dots, X_n)$ r.v. in $\mathbb{R}^n)$

$(X_1, \dots, X_n$ indep.$) \iff \forall t \in \mathbb{R}^n$ $\varphi_X(t) = \varphi_{X_1}(t_1) \times \dots \times \varphi_{X_n}(t_n)$

Proof. $\boxed{\Rightarrow}$ $E(e^{i\langle t, X\rangle}) = E(e^{i(t_1 X_1 + \dots + t_n X_n)})$

$\overset{indep.}{=} E(e^{it_1 X_1}) \times \dots \times E(e^{it_n X_n})$.

$\boxed{\Leftarrow}$ Let $Z_1, \dots, Z_n$ indep. s.t. $\mu_{Z_i} = \mu_{X_i}$ for every i

We have $\varphi_X(t) = \varphi_Z(t)$ $\forall t \in \mathbb{R}^n$

Hence $\mu_X = \mu_Z$, which implies $\forall A_1, \dots, A_n \in \mathcal{B}(\mathbb{R})$

$P(X_1 \in A_1, \dots, X_n \in A_n) = P(Z_1 \in A_1, \dots, Z_n \in A_n) = \prod_i P(Z_i \in A_i) = \prod_i P(X_i \in A_i)$.

Application Let $X_1, \dots, X_n$ indep. $\mathcal{N}(0,1)$.

$$Y = \sum_{i=1}^{n} \lambda_i X_i \qquad \lambda_i \in \mathbb{R}$$

$$Z = \sum_{i=1}^{n} \mu_i X_i \qquad \mu_i \in \mathbb{R}$$

$$Y, Z \text{ indep.} \iff E(YZ) = E(Y)E(Z)$$

Proof. $\boxed{\Leftarrow}$ $\varphi_Y(t) = \exp\left(-\frac{1}{2}\sum \lambda_j^2 t^2\right)$

$$\varphi_Z(t) = \exp\left(-\frac{1}{2}\sum \mu_j^2 t^2\right)$$

$$E(YZ) = \sum_{j=1}^{n} \lambda_j \mu_j \overset{\text{Hyp.}}{=} 0 \qquad \left(E(Y) = 0\right)$$

$$\varphi_{(Y,Z)}(t_1, t_2) = E\left(e^{i\, t_1 Y + t_2 Z}\right.$$

$$= E\left(e^{i \sum_{j=1}^{n} (t_1 \lambda_j + t_2 \mu_j) X_j}\right)$$

$$= \exp\left(-\frac{1}{2}\sum_{j=1}^{n} (t_1 \lambda_j + t_2 \mu_j)^2\right)$$

$$= \exp\left(-\frac{1}{2}\sum_{j=1}^{n} \lambda_j^2 t_1^2 - \frac{1}{2}\sum_{j=1}^{n} \mu_j t_2^2 - \underbrace{\sum_j \lambda_j \mu_j t_1 t_2}_{=0}\right)$$

$$= \varphi_Y(t_1)\varphi_Y(t_2) \qquad \blacksquare$$