CHAPTER 6

WEAK CONVERGENCE OF PROBABILITY MEASURES.

Goals : • gives sense to $\mu_n \longrightarrow \mu$ for probability measures.

• present the main characterizations needed for cv in distribution of r.v.

Setup $(\Omega, \mathcal{F}, P)$ fixed proba. space.

(E, d) metric space

$\mathcal{B}(E)$ Borel σ-algebra $(\mathcal{B}(E) = \sigma(\{O,\ O \text{ open set in } E\}))$

1 INTRODUCTION.

$(\mu_n)_{n \geq 1}, \mu$ sequence of probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$

Goal: How can one give sense to $\mu_n \xrightarrow[n\to\infty]{} \mu$?

Example 1 : . $\mu_n = p_n \delta_1 + (1-p_n)\delta_0$ (law of $X_n \sim \text{Ber}(p_n)$)

with $p_n \xrightarrow[n\to\infty]{} \frac{1}{2}$

We would like $\mu_n \longrightarrow \frac{1}{2}\delta_1 + \frac{1}{2}\delta_0$

Example 2 : $\delta_{\frac{1}{n}}$ (law of $X_n = \frac{1}{n}$ a.s.).

We would like $\delta_{\frac{1}{n}} \xrightarrow[n\to\infty]{} \delta_0$

First idea:

Def. $\mu_n \xrightarrow{(s)} \mu$ if $\forall A \in \mathcal{B}(\mathbb{R})$ $\mu_n(A) \to \mu(A)$

↳ Exple 1 $\mu_n(A) = p_n \mathbb{1}_{1\in A} + (1-p_n)\mathbb{1}_{0\in A}$

$$\xrightarrow[n\to\infty]{} \frac{1}{2}\mathbb{1}_{1\in A} + \frac{1}{2}\mathbb{1}_{0\in A} = \mu(A) \quad \text{ok}$$

↳ Exple 2 $A = \{0\}$

$$\forall n\geq 1 \ \delta_{\frac{1}{n}}(A) = 0 \ \cancel{\xrightarrow[n\to\infty]{}} \ \delta_0(A) = 1.$$

Hence $\delta_{\frac{1}{n}} \ \cancel{\xrightarrow{(s)}} \ \delta_0$

2 WEAK CONVERGENCE OF PROBABILITY MEASURES.

Not.: $\mathcal{C}_b = \{ f : E \to \mathbb{R} \text{ continuous bounded} \}$

Def. Let $\mu_n, n\geq 1$ and μ probability measures. We say that $(\mu_n)_n$ cv weakly to μ ($\mu_n \xrightarrow{(w)} \mu$) if

$$\boxed{\forall f \in \mathcal{C}_b \quad \int f \, d\mu_n \xrightarrow[n\to\infty]{} \int f d\mu}.$$

Rk: Since f continuous it is measurable, and

$$\int_E |f| d\mu \leq \|f\|_\infty \underbrace{\int_E 1 \, d\mu}_{\mu(E)} < \infty$$

Hence $\int f d\mu_n$, $\int f d\mu$ are well-defined

Ex 1 : $\mu_n = p_n \delta_1 + (1-p_n)\delta_0 \qquad \mu = \frac{1}{2}\delta_1 + \frac{1}{2}\delta_0 \quad \left(p_n \to \frac{1}{2}\right)$

$$\int f \, d\mu_n = p_n f(1) + (1-p_n) f(0) \underset{n\to\infty}{\longrightarrow} \frac{1}{2} f(1) + \frac{1}{2} f(0) = \int f \, d\mu$$

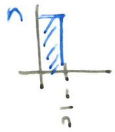
CCL: $\mu_n \xrightarrow{(w)} \mu$.

Ex 2 $\int f \, d\delta_{\frac{1}{n}} = f\left(\frac{1}{n}\right) \underset{n\to\infty}{\longrightarrow} f(0) = \int f d\delta_0$

because f continuous.

CCL: $\delta_{\frac{1}{n}} \xrightarrow{(w)} \delta_0$.

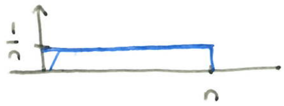
Ex 3 $\mu_n = n \times \mathrm{Leb}_{[0,\frac{1}{n}]}$



$$\int f \, d\mu_n = n \int_0^{1/n} f(t)\, dt \underset{n\to\infty}{\longrightarrow} f(0)$$

CCL $\mu_n \xrightarrow{(w)} \delta_0$

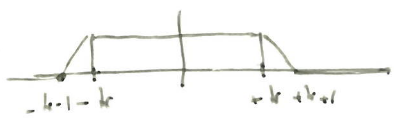
Ex 4 $\mu_n = \frac{1}{n} \mathrm{Leb}_{[0,n)}$



(Pro of $X \sim \mathcal{U}([0,n])$)

Assume $\mu_n \to \mu$ proba measure

Let $f_k =$



$$\mu([-k,k]) \leq \int f_k \, d\mu = \lim_{n\to\infty} \underbrace{\int f_k \, d\mu_n}_{\leq \mu_n([0,k+1])} = 0$$

Hence $\mu(\mathbb{R}) = 0$ contradiction.

CCL: $(\mu_n)_n$ does not converge weakly

Ex 5 $(\delta_n)_{n\geq 1}$ does not converge weekly (exercise)

2 HAUSDORF PROPERTY OF WEAK CONVERGENCE TOPOLOGY.

Prop. Let μ, ν be two probability measures on E.

If
$$\forall f \in \mathcal{C}_b \quad \int f \, d\mu = \int f \, d\nu,$$
then $\mu = \nu$.

Proof. Let $F \subset E$ be a closed set. For every $k \geq 1$, consider the continuous function def. by

$$\forall x \in E \quad f_k(x) = \max\left(0, 1 - k\, d(x, F)\right).$$

For every k, we have
$$\underbrace{\int f_k \, d\mu}_{\substack{\downarrow \text{dom. cv.} \\ \mu(F)}} = \underbrace{\int f_k \, d\nu}_{\substack{\downarrow \\ \nu(F)}}$$

Hence μ coincides with ν on the π-system $\{F : F \text{ closed}\}$ generating $\mathcal{E}$. Therefore $\mu = \nu$.

Corollary. Let $\mu_n, n \geq 1$, μ, ν proba measures on E.
If $\mu_n \xrightarrow{(w)} \mu$ and $\mu_n \xrightarrow{(w)} \nu$, then $\mu = \nu$.

Proof. $\forall f \in \mathcal{C}_b$, we have $\int f \, d\mu = \lim_{n \to \infty} \int f \, d\mu_n = \int f \, d\nu$.

Hence $\mu = \nu$.

3 PORTMANTEAU THEOREM

Motivation: relation between $\mu_n(A)$ and $\mu(A)$?

Thm: Let $\mu_n, n \geq 1$, μ probability measures on E. The following are equivalent:

(i) $\mu_n \xrightarrow{(w)} \mu$.

(ii) $\forall F \subset E$ closed $\limsup \mu_n(F) \leq \mu(F)$

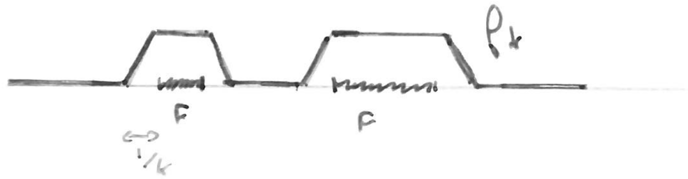
(iii) $\forall O \subset E$ open $\liminf \mu_n(O) \geq \mu(O)$

(iv) $\forall B \in \mathcal{B}(E)$ s.t. $\underset{\overline{B} \setminus \mathring{B}}{\underset{\uparrow}{\mu(\partial B)}} = 0$ $\mu_n(B) \underset{n \to \infty}{\longrightarrow} \mu(B)$

Proof. (i) $\Rightarrow$ (ii)

Let F be a closed set. For every $k \geq 1$ consider the continuous function $f_k : E \to [0,1]$ def. by

$$\forall x \in E \quad f_k(x) = \max(0, 1 - k\, d(x, F)).$$



$\forall k \ \forall n$

$$\mu_n(F) = \int 1_F \, d\mu_n \leq \int f_k \, d\mu_n$$

Hence taking the limsup as $n \to \infty$, we get

$$\forall k \geq 1 \quad \limsup_{n \to \infty} \mu_n(F) \leq \underbrace{\int f_k \, d\mu}_{\substack{\xrightarrow[k \to \infty]{} \mu(F) \\ \text{dominated} \\ \text{c.v}}}$$

(ii) $\Leftrightarrow$ (iii)

$\forall F \subset E \quad (F \text{ closed}) \Leftrightarrow (O = F^c \text{ open}) \qquad \mu.$

(ii) $\Leftrightarrow \forall F$ closed $\limsup \mu_n(F) \leq \mu(F)$

$\Leftrightarrow \forall F$ closed $\liminf \underbrace{(1-\mu_n(F))}_{\mu_n(F^c)} \geq \underbrace{1-\mu(F)}_{\mu(F^c)}$

$\Leftrightarrow \forall O$ open $\liminf \mu_n(O) \geq \mu(O) \Leftrightarrow$ (iii)

(ii), (iii) $\Rightarrow$ (iv.)

Let $B \in \mathcal{B}(E)$

$$\mu(\mathring{B}) \overset{(iii)}{\leq} \liminf \mu_n(\mathring{B}) \leq \liminf \mu_n(B) \leq \limsup \mu_n(B) \leq \limsup \mu_n(\bar{B}) \overset{(ii)}{\leq} \mu(\bar{B})$$

If $\mu(\bar{B} \setminus \mathring{B}) = 0$, then $\mu(\bar{B}) = \mu(\mathring{B})$ and all the inequalities above are equalities.

(iv) $\Rightarrow$ (i)

Let $f \in \mathcal{C}_b$. WLOG assume $0 \leq f \leq 1$

$$\int_E f \, d\mu = \int_E \left(\int_0^1 1_{t \leq f(x)} \, dt \right) d\mu(x)$$

$$\overset{\text{Fub}}{=} \int_0^1 \left(\int_E 1_{t \leq f(x)} \, d\mu(x) \right) dt$$

$$= \int_0^1 \mu(\{x \in E : f(x) \geq t\}) \, dt$$

$\left(\text{Rk: } E(f(X)) = \int_0^\infty P(f(X) \geq t) \, dt. \right)$

Set $A_t := \{x \in E : f(x) \geq t\} \qquad t \geq 0$

Claim : $D := \{ t \in [0,1] \quad \mu(\partial A_t) > 0 \}$ is at most countable.

Proof of the claim:

Let $t \in [0,1]$. $\overline{A_t} = A_t$ because A_t is closed, and $\{x \in E : f(x) > t\} \subset \overset{\circ}{A_t}$. Hence

$$\partial A_t \subset \{x \in E : f(x) = t\}$$

For every $k \geq 1$ $\left| \{ t : \mu(\partial A_t) \geq \frac{1}{k} \} \right| \leq k$

(otherwise $\exists t_1, \dots, t_{k+1}$ distinct $\mu(\partial A_{t_i}) \geq \frac{1}{k}$, which contradicts $\mu(\partial A_{t_1}) + \dots + \mu(\partial A_{t_{k+1}}) \leq 1$)

Hence $\{ t : \mu(\partial A_t) > 0 \} = \bigcup_{k=1}^{\infty} \{ t : \mu(\partial A_t) \geq \frac{1}{k} \}$ at most countable. ∎

For every $n \geq 1$, we have

$$\left| \int f \, d\mu_n - \int f \, d\mu \right| = \left| \int_0^1 (\mu_n(A_t) - \mu(A_t)) \, dt \right|$$

$$\leq \int_0^1 \underbrace{|\mu_n(A_t) - \mu(A_t)| \mathbb{1}_{t \notin D}}_{\xrightarrow[n \to \infty]{} 0} \, dt \quad \text{(by the claim)}$$

Hence, by dominated convergence

$$\int f \, d\mu_n \xrightarrow[n \to \infty]{} \int f \, d\mu.$$ ∎

<u>Example</u>: $\mu_n = \delta_{\frac{1}{n}}$ $\quad \mu = \delta_0$

- $F = (-\infty, 0]$ closed $\quad \mu_n(F) = 0 \leqslant \mu(F) = 1.$
- $O = (0, +\infty)$ open $\quad \mu_n(O) = 1 \geqslant \mu(O) = 0.$
- $\forall A \in \mathcal{B}(\mathbb{R}) : 0 \notin \partial A \quad \mu_n(A) \longrightarrow \mu(A)$

$A = [-1, 0]$

mass enters in A at the limit.

$A = (-1, \varepsilon)$

mass enters in A before the limit.

4 RESTRICTING THE SET OF TEST FUNCTIONS IN $\mathbb{R}^d$.

<u>Motivation</u>: $\mathcal{C}_b$ is large $\to$ proving "$\forall f \in \mathcal{C}_b \dots$" may be hard.

Setup of the section : $E = \mathbb{R}^d$, $\cdot$ Euclidean distance.

$$\mathcal{C}_c = \{ f : \mathbb{R}^d \to \mathbb{R} \text{, continuous , compact support} \}$$

$$\mathcal{C}_c^\infty = \{ f : \mathbb{R}^d \to \mathbb{R} \text{, infinitely diff. , compact support} \}$$

<u>Rk</u>: $\mathcal{C}_c^\infty \subset \mathcal{C}_c \subset \mathcal{C}_b$.

Thm. Let $\mu_n, n \geq 1$, μ probability measures on $\mathbb{R}^d$.
The following are equivalent:

(i) $\mu_n \xrightarrow{(w)} \mu$.

(ii) $\forall f \in \mathcal{C}_c \quad \int f \, d\mu_n \xrightarrow[n \to \infty]{} \int f \, d\mu$.

(iii) $\forall f \in \mathcal{C}_c^\infty \quad \int f \, d\mu_n \xrightarrow[n \to \infty]{} \int f \, d\mu$.

Proof: (i) ⇒ (ii) ⇒ (iii) follows from $\mathcal{C}_c^\infty \subset \mathcal{C}_c \subset \mathcal{C}_b$

(iii) ⇒ (i) : Let $f \in \mathcal{C}_b$. Let $\varepsilon > 0$.

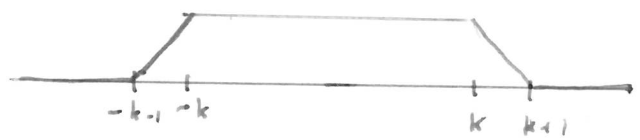
Pick $k \geq 1$ large enough such that

$$\mu([-k, k]^d) > 1 - \frac{\varepsilon}{\|f\|_\infty}$$

(well def. because $\lim_{\ell \to \infty} \mu([-\ell, \ell]^d) = \mu(\mathbb{R}^d) = 1$)

Let χ be a continuous function s.t.

$$\chi(x) = \begin{cases} 0 & \text{if } x \notin [-k-1, k+1]^d, \\ 1 & \text{if } x \in [-k, k]^d. \end{cases}$$



$$\int_{\mathbb{R}^d} (1-\chi) \, d\mu \leq \int_{\mathbb{R}^d} 1_{\mathbb{R}^d \setminus [-k,k]^d} \, d\mu$$

$$= \mu(\mathbb{R}^d \setminus [-k, k]^d) < \frac{\varepsilon}{\|f\|_\infty}$$

Since $\chi \in \mathcal{C}_c$, we have

$$\lim_{n\to\infty}\left(\int_{\mathbb{R}^d}(1-\chi)\,d\gamma_n\right) = 1 - \lim_{n\to\infty}\int \chi\,d\gamma_n$$

$$= 1 - \int \chi\,d\gamma$$

$$= \int (1-\chi)\,d\gamma. < \frac{\varepsilon}{2\|f\|_\infty}.$$

$$\left|\int f\,d\gamma_n - \int f\,d\gamma\right|$$

$$\leq \underbrace{\left|\int f(1-\chi)\,d\gamma_n\right|}_{\substack{\leq \|f\|_\infty \int(1-\chi)d\gamma_n \\ \leq \varepsilon/2 \\ \text{for } n \text{ large}}} + \underbrace{\left|\int f\chi\,d\gamma_n - \int f\chi\,d\gamma\right|}_{\xrightarrow[n\to\infty]{} 0} + \underbrace{\left|\int f(1-\chi)\,d\gamma\right|}_{\substack{\leq \|f\|_\infty \int(1-\chi)\,d\gamma \\ \leq \varepsilon/2}}$$

Hence $\limsup_{n\to\infty}\left|\int f\,d\gamma_n - \int f\,d\gamma\right| \leq \varepsilon$.

(iii) $\Rightarrow$ (ii) Let $f \in \mathcal{C}_c$. Let $\varepsilon > 0$. By Stone-Weierstrass thm, we can fix $g \in \mathcal{C}_c^\infty$ s.t

$$\|f-g\|_\infty \leq \frac{\varepsilon}{2}$$

$$\left|\int f\,d\gamma_n - \int f\,d\gamma\right| \leq \underbrace{\left|\int (f-g)\,d\gamma_n\right|}_{\substack{\leq \|f-g\|_\infty \\ \leq \varepsilon/2}} + \underbrace{\left|\int g\,d\gamma_n - \int g\,d\gamma\right|}_{\xrightarrow[n\to\infty]{} 0} + \underbrace{\left|\int (f-g)\,d\gamma\right|}_{\substack{\leq \|f-g\|_\infty \\ \leq \frac{\varepsilon}{2}}}.$$

Hence $\limsup_{n\to\infty} \left|\int f\,d\gamma_n - \int f\,d\gamma\right| \leq \varepsilon$ ∎

Reminders: Transfert formula.

$(\Omega, \mathcal{F}, P)$ proba space.

$(E, \mathcal{E})$ measurable space.

X r.v in E $\quad (X : \Omega \to E$ measurable$)$

Law of X: μ proba measure on E.

$\forall B \in \mathcal{E} \quad \mu(B) = P(X \in B)$

Transfert formula:

. $f : E \to \mathbb{R}$ meas. ≥ 0

$$\boxed{E(f(X)) = \int_E f \, d\mu}$$

Appli. - E finite or countable

$$\mu = \sum_{x \in E} p_x \delta_x \qquad p_x = P(X = x)$$

$$\int f \, d\mu = \sum f(x) P(X = x).$$

- $E = \mathbb{R}^d \qquad d\mu = g \, d\,\mathrm{Leb}_{\mathbb{R}^d} \qquad g \geq 0$

$$\int f \, d\mu = \int_{\mathbb{R}^d} f(x) \cdot g(x) \, dx.$$

<u>Proof.</u> $E(f(X)) = \int_\Omega \underbrace{f(X)}_{f\circ X} \, dP \overset{(*)}{=} \int_E f \, d\mu$

$\uparrow$
abstract
change of variable
$x = X(\omega)$
$d\mu = d(X \# P)$

$\hookrightarrow$ proof of $(*)$ ① $f = 1_A$

② $f = \sum \lambda_i 1_{A_i}$ linearity

③ f meas. ≥ 0 Monotone cv. $\blacksquare$

<u>Rk:</u> Extends to $f \in L^1$.