CHAPTER 7

CONVERGENCE IN DISTRIBUTION.

Goals: • relation cv in distribution / weak cv. of measures.

• relation / differences with other types of cv.

• proof of CLT.

Setup: (E, d) metric space

$\mathcal{C}_b = \{ f : E \to \mathbb{R} \text{ continuous bounded} \}$

$(\Omega, \mathcal{F}, P)$ fixed proba. space.

I DEFINITION

Def. Let $X_n, n \geq 1, X$ r.v.s with values in E. We say that (X_n) cv in distribution towards X (written $X_n \xrightarrow{(d)} X$) if

$$\forall f \in \mathcal{C}_b \quad E(f(X_n)) \xrightarrow[n\to\infty]{} E(f(X)).$$

Rk. $X_n \xrightarrow{(d)} X \underset{\text{(transfert formula)}}{\iff} \forall f \in \mathcal{C}_b \int f \, d\mu_{X_n} \longrightarrow \int f \, d\mu$

$$\iff \mu_{X_n} \xrightarrow{(w)} \mu_X .$$

Ex 1: $X_n \sim Ber(p_n)$ $\quad p_n \to \frac{1}{2}$

$$X_n \xrightarrow[n\to\infty]{(d)} X \quad \text{where } X \sim Ber\left(\tfrac{1}{2}\right).$$

Ex 2: $X_n = \frac{1}{n}$ a.s.

$$X_n \xrightarrow{(d)} X \quad \text{where } X = 0 \text{ a.s.}$$

Ex 3: $X_n \sim \mathcal{U}([0, \frac{1}{n}])$

$$X_n \xrightarrow{(d)} X \quad \text{where } X = 0 \text{ a.s.}$$

Ex 4: $X_n \sim \mathcal{U}([0, n])$ does not converge in distribution.

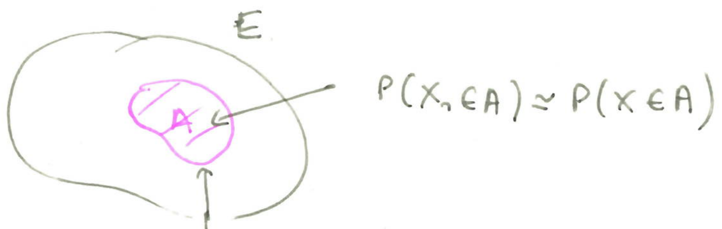
Ex 5: $X_n = n$ does not converge in distribution.

Rk: let $X_n, n \geq 1$, X r.v.s in E

$$X_n \xrightarrow{(d)} X \iff \forall F \subset E \text{ closed} \quad \limsup P(X_n \in F) \leq P(X \in F)$$

$$\iff \forall O \subset E \text{ open} \quad \liminf P(X_n \in O) \geq P(X \in O)$$

$$\iff \forall A \text{ s.t. } P(X \in \partial A) = 0 \quad \lim P(X_n \in A) = P(X \in A).$$



if X gives No mass at ∂A

2 CHARACTERIZATION IN $\mathbb{R}$.

$E = \mathbb{R}$ $\left(d(x,y) = |x-y| . \right)$

Thm. Let X_n, $n \geq 1$, X real r.v.s. The following are equivalent.

(i) $X_n \xrightarrow{(d)} X$.

(ii) $F_{X_n}(t) \xrightarrow[n\to\infty]{} F_X(t)$ for every t continuity point of F_X.

Reminder: If X real r.v., we define its cdf (cumulative distribution fct) by

$$\forall t \in \mathbb{R} \quad F_X(t) = P(X \leq t).$$

- F_X is continuous at $t \iff P(X = t) = 0$.
- The number of discontinuity points of F_X is at most countable. ($\{$continuity points$\}$ is dense in $\mathbb{R}$)

Proof of Thm.

(i) $\Rightarrow$ (ii) Let $t \in \mathbb{R}$ be a continuity point of F_X.

$$F_{X_n}(t) = P\left(X_n \in (-\infty, t] \right)$$

$$\xrightarrow[n\to\infty]{} P\left(X \in (-\infty, t] \right) = F_X(t).$$

Portmanteau Thm (because $P(X = t) = 0$)

(ii) $\Rightarrow$ (i) We will prove that

$$\forall O \subset E \text{ open } \liminf_{n\to\infty} P(X_n \in O) \geq P(X \in O).$$

Step 1: $O = (a,b)$ $a < b$ $a \in \{-\infty\} \cup \mathbb{R}$, $b \in \mathbb{R} \cup \{+\infty\}$.

(O = open interval)

Let $a_k \downarrow a$ and $b_k \uparrow b$ s.t.

$\forall k$ a_k, b_k continuity points of F_X.

For $k \geq 1$, $n \geq 1$, we have

$$P(X_n \in (a,b)) \geq P(X_n \in (a_k, b_k])$$

$$\left(= F_{X_n}(b_k) - F_{X_n}(a_k)\right)$$

Hence for $k \geq 1$

$$\liminf_{n\to\infty} P(X_n \in (a,b)) \geq P(X \in (a_k, b_k]).$$

Taking the limit as $k \to \infty$, we get

$$\liminf P(X_n \in (a,b)) \geq P(X \in (a,b)).$$

Step 2: O general open set. There exist $(I_j)_{j\in J}$ open disjoint intervals s.t

$$O = \bigcup_{j\in J} I_j$$ and J at most countable.

$$\forall n \geq 1. \quad P(X_n \in O) = \sum_{j \in J} P(X_n \in I_j).$$

Hence, by Fatou's lemma,

$$\liminf P(X_n \in O) \geq \sum_{j \in J} \liminf P(X_n \in I_j)$$

$$\overset{\text{Step 1}}{\geq} \sum_{j \in J} P(X \in I_j)$$

$$= P(X \in O). \quad \blacksquare$$

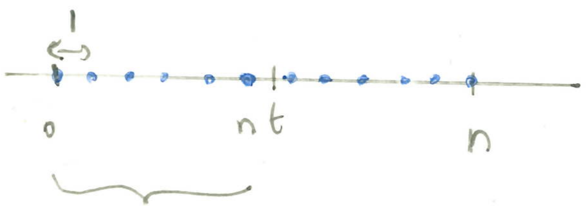
Application $\quad X_n \sim \mathcal{U}(\{0, \frac{1}{n}, \dots, \frac{n-1}{n}, 1\}) \quad \left(P(X_n = \frac{k}{n}) = \frac{1}{n+1}\right)$

$X \sim \mathcal{U}([0,1])$

$X_n \xrightarrow{(d)} X$

$\forall t \in [0,1]$

$$F_{X_n}(t) = \frac{\lfloor nt \rfloor + 1}{n+1} \xrightarrow[n \to \infty]{} t$$



$\lfloor nt \rfloor + 1$ points

Application:

Let $\lambda > 0$

- $X_n \sim Bin\left(n, \frac{\lambda}{n}\right) \qquad n \geq \lambda .$

For $k \in \mathbb{N}$ fixed

$$P(X_n = k) = \binom{n}{k}\left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k}$$

$$= \frac{\lambda^k}{k!} \times \underbrace{\frac{n \times \cdots \times n-k+1}{n^k}}_{\xrightarrow[n\to\infty]{} 1} \times \underbrace{\left(1 - \frac{\lambda}{n}\right)^{n-k}}_{\xrightarrow[n\to\infty]{} e^{-\lambda}}$$

$$\longrightarrow \frac{\lambda^k}{k!} e^{-\lambda}.$$

$X_n \xrightarrow{(d)} X \qquad$ where $X \sim Poi(\lambda)$

$n >> 1$

$X_n \sim Bin\left(n, \frac{\lambda}{n}\right) \simeq Poi(\lambda)$

n trials with success probability $\frac{\lambda}{n}$.

3 RESTRICTING THE CLASS OF TEST FUNCTIONS

$E = \mathbb{R}^d \qquad \mathcal{C}_c = \{ f : \mathbb{R}^d \to \mathbb{R} \text{ continuous, compact support} \}$

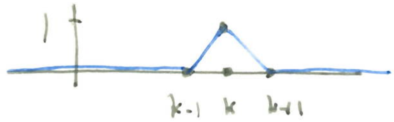
$$\boxed{X_n \xrightarrow{(d)} X \iff \forall f \in \mathcal{C}_c \quad E(f(X_n)) \longrightarrow E(f(X)).}$$

(Chap 6)

<u>Application</u> Assume $X_n \in \mathbb{Z}$ a.s. $X \in \mathbb{Z}$ a.s.

$$X_n \xrightarrow{(d)} X \iff \forall k \in \mathbb{Z} \quad P(X_n = k) \xrightarrow[n\to\infty]{} P(X = k)$$

<u>Pf:</u> $\boxed{\Rightarrow}$ Let $k \in \mathbb{Z}$ $\quad f_k =$



$$P(X = k) = E(f_k(X)) = \lim_{n\to\infty} E(f_k(X_n)) =$$

$$= \lim_{n\to\infty} P(X_n = k).$$

$\boxed{\Leftarrow}$ Let $f \in \mathcal{C}_c$. Let $a \in \mathbb{N}$ s.t. $f(x) = 0 \ \forall |x| > a$.

$$E(f(X)) = \sum_{k=-a}^{a} f(k)\, P(X = k)$$

$$= \lim_{n\to\infty} \sum_{k=-a}^{a} f(k)\, P(X_n = k)$$

$$= \lim_{n\to\infty} E(f(X_n)).$$

4 FOURIER CHARACTERIZATION

$E = \mathbb{R}^d$. $d \geq 1$ $\qquad \mathcal{C}_c^\infty = \{ f : \mathbb{R}^d \to \mathbb{R}$ infinitely diff., compact support $\}$.

Thm (Levy)

Let X_n, $n \geq 1$ X r.v.s in $\mathbb{R}^d$. The following are equivalent:

(i) $X_n \xrightarrow{(d)} X$

(ii) $\forall t \in \mathbb{R}^d$ $\varphi_{X_n}(t) \xrightarrow[n\to\infty]{} \varphi_X(t)$.

Proof. $\boxed{\Rightarrow}$ Let $t \in \mathbb{R}^d$

$$e^{itx} = \underbrace{\cos(t,x)}\ + i\ \underbrace{\sin(tx)}$$

continuous in x, bounded

Hence $E(e^{itX_n}) \xrightarrow[n\to\infty]{} E(e^{itX})$.

$\boxed{\Leftarrow}$ Let $f \in \mathcal{C}_c^\infty$ (in particular $\hat{f} \in L^1$)

$$E(f(X_n)) = \frac{1}{(2\pi)^d} \int \hat{f}(t) \overline{\varphi_{X_n}(t)}\, dt$$

$$\xrightarrow[n\to\infty]{} \frac{1}{(2\pi)^d} \int \hat{f}(t) \overline{\varphi_X(t)}\, dt$$

(dominated cv: $|\hat{f}\, \overline{\varphi_{X_n}}| \leq |\hat{f}|$)

$$= E(f(X)) \qquad \blacksquare$$

5 CENTRAL LIMIT THEOREM

Thm. Let $X_1, X_2, \dots$ iid $E(X_1^2) < \infty$

$m = E(X_1)$ $\sigma^2 = Var(X_1)$. Writing

$S_n = X_1 + \dots + X_n$, we have

$$\boxed{\frac{S_n - nm}{\sqrt{n\sigma^2}} \xrightarrow{(d)} Z},$$

where $Z \sim \mathcal{N}(0,1)$.

Proof. Let $Y_n = \dfrac{X_n - m}{\sigma}$ for every $n \geq 1$.

$$Z_n = \frac{Y_1 + \dots + Y_n}{\sqrt{n}} = \frac{S_n - nm}{\sqrt{n}}.$$

We have $Y_1 \in L^2$ $E(Y_1) = 0$ $E(Y_1^2) = 1$.

Hence $\varphi = \varphi_{Y_1}$ is $\mathcal{C}^2$ and

$$\forall u \in \mathbb{R} \quad \varphi(u) = 1 - \frac{u^2}{2} + o(u^2)$$

Fix $t \in \mathbb{R}$. By independence of the $(Y_n)_{n \geq 1}$,

$$\varphi_{Z_n}(t) = E\left(e^{it\frac{Y_1 + \dots + Y_n}{\sqrt{n}}}\right)$$

$$= \varphi\left(\frac{t}{\sqrt{n}}\right)^n$$

$$\xrightarrow[n \to \infty]{} e^{-\frac{t^2}{2}} = \varphi_Z(t)$$

$$! \; \varphi\left(\frac{t}{\sqrt{n}}\right) = 1 - \frac{t^2}{2n} + \underset{n \to \infty}{o}\left(\frac{1}{n}\right)$$

6 LINDENBERG PROOF OF CLT.

$E = \mathbb{R}$.

Lemma. Let $f \in \mathcal{C}_c^\infty$. For $y, h \in \mathbb{R}$, let

$$g(y,h) = f(y+h) - f(y) - f'(y) \cdot h - f''(y) \frac{h^2}{2}.$$

There exists a constant C s.t.

$$\forall y, h \in \mathbb{R} \quad |g(y,h)| \leq C \left(h^2 \underset{\text{min}}{\wedge} |h|^3 \right).$$

Proof. Let $y, h \in \mathbb{R}$. By Taylor identity, there exists $a, b \in [y, y+h]$ ($[y+h, y]$ if $h < 0$) s.t.

$$\text{and} \begin{cases} f(y+h) = f(y) + f'(y) \cdot h + f''(a) \cdot \frac{h^2}{2}, \\ f(y+h) = f(y) + f'(y) h + f''(y) \frac{h^2}{2} + f'''(b) \frac{h^3}{3!}. \end{cases}$$

The first identity gives

$$|g(y,h)| = \left| f''(a) \cdot \frac{h^2}{2} - f''(y) \frac{h^2}{2} \right|$$
$$\leq \|f''\|_\infty \times h^2$$

The second identity gives

$$|g(y,h)| = \left| f'''(b) \frac{h^3}{3!} \right| \leq \frac{1}{3!} \|f'''\|_\infty \cdot |h|^3.$$

This concludes the proof with $C = \max\left(\|f''\|_\infty, \frac{\|f'''\|_\infty}{3!} \right)$.

Let X be a r.v. with $E(X)=0$ $E(X^2)=1$.

Let $X_1, X_2, X_3, \ldots$ be iid copies of X.

Our goal is to give an alternative proof of

$$\frac{X_1+\cdots+X_n}{\sqrt{n}} \xrightarrow{(d)} Z \text{ where } Z \sim \mathcal{N}(0,1).$$

Let $f \in \mathcal{C}_c^\infty$. Let $n \geq 1$ and $Z_1, \ldots, Z_n$ iid $\mathcal{N}(0,1)$, indep. of $(X_i)_{i\geq 1}$.

Key property of $\mathcal{N}(0,1)$:

$$Z \overset{(d)}{=} \frac{Z_1+\cdots+Z_n}{\sqrt{n}},$$

$$E\left(f\left(\frac{X_1+\cdots+X_n}{\sqrt{n}}\right)\right) - E(f(Z)) = E\left(f\left(\frac{X_1+\cdots+X_n}{\sqrt{n}}\right) - f\left(\frac{Z_1+\cdots+Z_n}{\sqrt{n}}\right)\right)$$

$$= \sum_{i=1}^n E\left(f\left(\frac{X_1+\cdots+X_i+Z_{i+1}+\cdots+Z_n}{\sqrt{n}}\right) - f\left(\frac{X_1+\cdots+X_{i-1}+Z_i+\cdots+Z_n}{\sqrt{n}}\right)\right)$$

$$= \sum_{i=1}^n E\left(f\left(Y_i+\frac{X_i}{\sqrt{n}}\right) - f\left(Y_i+\frac{Z_i}{\sqrt{n}}\right)\right),$$

where $Y_i := \dfrac{X_1+\cdots+X_{i-1}+Z_{i+1}+\cdots+Z_n}{\sqrt{n}}$.

Fix $i \in \{1,\ldots,n\}$. Let g as in the lemma, we have.

$$f\left(Y_i+\frac{X_i}{\sqrt{n}}\right) - f\left(Y_i+\frac{Z_i}{\sqrt{n}}\right) = f'(Y_i)\times\left(\frac{X_i}{\sqrt{n}}-\frac{Z_i}{\sqrt{n}}\right) + f''(Y_i)\left(\frac{X_i^2}{2}-\frac{Z_i^2}{2}\right)$$

$$+\, g(Y_i, X_i) - g(Y_i, Z_i)$$

Since X_i, Y_i, Z_i are independent (by grouping), $E(X_i) = E(Z_i)$ and $E(X_i^2) = E(Z_i^2)$, taking the expectation above gives

$$\left| E\left(f\left(Y_i + \frac{X_i}{\sqrt{n}}\right) - f\left(Y_i + \frac{Z_i}{\sqrt{n}}\right)\right)\right| = \left| E\left(g\left(Y_i, \frac{X_i}{\sqrt{n}}\right) - g\left(Y_i, \frac{Z_i}{\sqrt{n}}\right)\right)\right|$$

$$\leq E\left(\left|g\left(Y_i, \frac{X_i}{\sqrt{n}}\right)\right|\right) + E\left(\left|g\left(Y_i, \frac{Z_i}{\sqrt{n}}\right)\right|\right)$$

$$\leq C\, E\left(\frac{X_i^2}{n} \wedge \frac{|X_i|^3}{n^{3/2}}\right) + C\, E\left(\frac{|Z_i|^3}{n^{3/2}}\right)$$

$$\leq \frac{C}{n} E\left(X^2 \wedge \frac{|X|^3}{\sqrt{n}}\right) + \frac{3C}{n^{3/2}}.$$

since $Z_i \sim \mathcal{N}(0,1)$

$E(|Z_i|^3) \leq E(Z_i^4)^{\frac{3}{4}} = 3^{3/4} \leq 3.$

Jensen.

<u>Conclusion:</u> For every $n \geq 1$

$$\left|E\left(f\left(\frac{X_1 + \dots + X_n}{\sqrt{n}}\right)\right) - E(f(Z))\right| \leq \underbrace{C\, E\left(X^2 \wedge \frac{|X|^3}{\sqrt{n}}\right)}_{\substack{\xrightarrow[n\to\infty]{} 0 \\ \text{(dominated cv.)}}} + \frac{3C}{\sqrt{n}} \xrightarrow[n\to\infty]{} 0$$

<u>Rk:</u> If $E(|X|^3) < \infty$, we get a $\frac{1}{\sqrt{n}}$ speed of cv in the CLT.