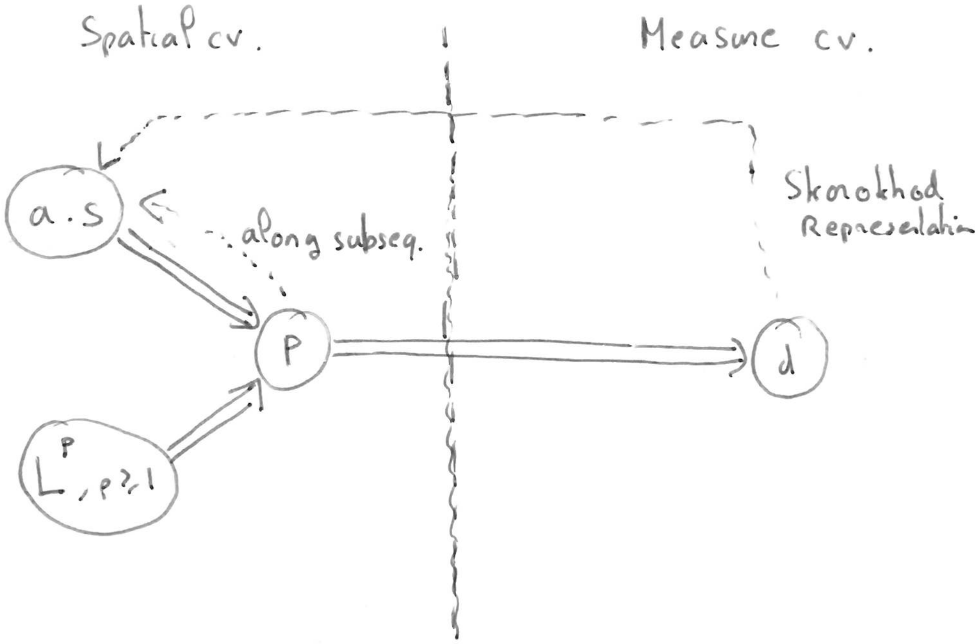
CHAPTER 8 :

PROPERTIES OF THE LIMITS.

<u>Goal:</u>
- Which properties of "standard" limits am I allowed to use when working with r.v.s ?
- Spatial limits vs measure limits : relation and differences



<u>Setup:</u> . $(\Omega, \mathcal{F}, P)$ fixed proba space.

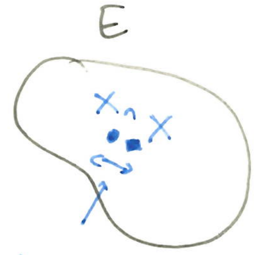
. (E, d) metric separable

"there exists $\{x_n\}_{n \in \mathbb{N}}$ dense in E."

1 INTRODUCTION

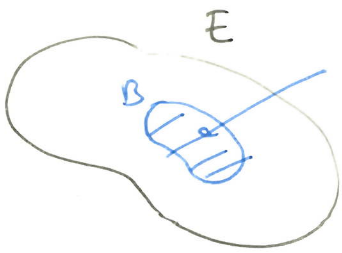
$$\left.\begin{array}{l} X_n \xrightarrow{P} X \\ X_n \xrightarrow{a.s.} X \\ X_n \xrightarrow{L^p} X \end{array}\right\}$$

different ways to give sense to: "X_n and X are typically close to each other"



$d(X_n, X)$ is "typically" small.

$X_n \xrightarrow{(d)} X$: X_n and X have similar probabilistic properties: if I sample X or X_n for n large, the two experiments will give similar outputs. But X_n does not need to be close to X (X_n does not even need to be defined on the same Ω).



Fixing a "regular B" X_n will have the same chance to lie in B.

Illustrative example: $X \sim \mathrm{Ber}\left(\frac{1}{2}\right)$

- $X_n := X$ $\quad X_n \xrightarrow{a.s.} X \quad X_n \xrightarrow{(d)} X$
- $Y_n := 1 - X$ $\quad Y_n \xrightarrow{(d)} X \quad (\mu_{Y_n} = \mu_X \ \forall n)$ but

Y_n is always far from X: $(Y_n \not\xrightarrow{a.s} X)$.

2 UNIQUENESS OF THE LIMIT

<u>Prop.</u> Let $X_n, n\geq 1, X, Y$ rvs in E.

$$\left.\begin{array}{l} X_n \xrightarrow{P} X \\ X_n \xrightarrow{P} Y \end{array}\right\} \Rightarrow X = Y \text{ a.s.}$$

<u>Proof.</u> Let $(X_{k(n)})$ be a subsequence s.t

$$X_{k(n)} \longrightarrow X \text{ a.s.}$$

Since $X_{k(n)} \xrightarrow{P} Y$, we can extract $X_{k(\ell(n))}$ s.t.

$$X_{k(\ell(n))} \longrightarrow Y \text{ a.s}.$$

Hence, almost surely, we have

$$X = \lim_{n\to\infty} X_{k(\ell(n))} = Y.$$

<u>Rk.</u> Let $X \sim \mathrm{Ber}(\frac{1}{2})$ $X_n = X$ $Y_n = 1-X$.

We have
$$\left.\begin{array}{l} X_n \xrightarrow{(d)} X \\ Y_n \xrightarrow{(d)} 1-X \end{array}\right\} \text{ but } X \neq 1-X.$$

For convergence in distribution, we have uniqueness of the law of the limit
$$\left.\begin{array}{l} X_n \xrightarrow{(d)} X \\ X_n \xrightarrow{(d)} Y \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} \mathbb{P}_{X_n} \xrightarrow{(w)} \mathbb{P}_X \\ \mathbb{P}_{X_n} \xrightarrow{(w)} \mathbb{P}_Y \end{array}\right. \Rightarrow \mathbb{P}_X = \mathbb{P}_Y.$$

3 IMAGE BY A CONTINUOUS FUNCTION.

Prop. Let (E', d') be a metric separable space.

$f : E \to E'$ continuous

$X_n, n \geq 1, X$ r.v.s in E. For $* = P$, a.s., (d).

$$X_n \xrightarrow{(*)} X \Rightarrow f(X_n) \xrightarrow{(*)} f(X)$$

Proof. $(*) =$ a.s. ok:

Let $A = \{ \omega \in \Omega ; X_n(\omega) \to X(\omega) \}$

We have $P(A) = 1$ and $\forall \omega \in A$

$$f(X_n(\omega)) \xrightarrow[n \to \infty]{} f(X(\omega)) .$$

$(*) = P$

Assume for contradiction $f(X_n) \underset{n \to \infty}{\overset{P}{\not\to}} f(X)$.

We can fix $\varepsilon > 0$ s.t.

$$P(d'(f(X_n), f(X)) > \varepsilon) \not\to 0$$

Consider $\delta > 0$ and a subsequence $X_{k(n)}$ s.t.

$$\forall n \quad P(d'(f(X_{k(n)}), f(X)) > \varepsilon) \geq \delta . \qquad (1)$$

We can extract $X_{k(\ell(n))} \xrightarrow{a.s} X$.

(because $X_{k(n)} \xrightarrow{P} X$) which implies $f(X_{k(\ell(n))}) \xrightarrow{a.s.} f(X)$,

and contradicts (1).

(*) $\Rightarrow$ (d)

Let $g : E' \to \mathbb{R}$ continuous bounded.

Since $g \circ f : E \to \mathbb{R}$ continuous bounded, we have

$$E\big(g(f(X_n))\big) \xrightarrow[n\to\infty]{} E\big(g(f(X))\big). \quad \blacksquare$$

4 OPERATIONS.

$E = \mathbb{R}^d$

Question

$$\left.\begin{array}{l} X_n \to X \\ Y_n \to Y \end{array}\right\} \; X_n + Y_n \to X + Y \; ?$$

ok for a.s. / L^p / P convergence

no in general for cv in distribution:

$X \sim \mathrm{Ber}\left(\frac{1}{2}\right) \qquad X_n = X \qquad Y_n = 1 - X$

$$\left\{\begin{array}{l} X_n \xrightarrow{(d)} X \\ Y_n \xrightarrow{(d)} X \end{array}\right. \qquad \text{but} \qquad X_n + Y_n \not\to 2X$$

Pb: The law of X_n and the law of Y_n do not determine the law of (X_n, Y_n).

<u>Prop.</u> Let $X_n, n \geq 1, X$ r.v.s in $\mathbb{R}^d$, $Y_n, n \geq 1, Y$ r.v.s in $\mathbb{R}^{d'}$. $* = P$ or a.s.

$$\left.\begin{array}{l} X_n \xrightarrow{*} X \\ Y_n \xrightarrow{*} Y \end{array}\right\} \iff (X_n, Y_n) \xrightarrow{(*)} (X, Y).$$

<u>Proof:</u> $*$ = a.s. ok.

$*$ = P. Let $\varepsilon > 0$

$$P\left(\|(X_n, Y_n) - (X, Y)\|_\infty \geq \varepsilon \right)$$

$$\leq P\left(\|X_n - X\|_\infty \geq \varepsilon \right) + P\left(\|Y_n - Y\|_\infty \geq \varepsilon \right) \xrightarrow[n \to \infty]{} 0$$

union bound

The property above allows us to make operations on limits:

If $f: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ continuous (e.g. $f(x, y) = x + y$)

$$\left.\begin{array}{l} X_n \xrightarrow{*} X \\ Y_n \xrightarrow{*} Y \end{array}\right\} \Rightarrow f(X_n, Y_n) \xrightarrow{*} f(X, Y).$$

For cv in distribution, the convergence of (X_n) and (Y_n) is not equivalent to convergence of (X_n, Y_n) in general. At the end of the chapter, we will discuss a particular case where it holds.

5 CONVERGENCE IN DISTRIBUTION V.S IN PROBABILITY.

Prop. Let X_n, $n \geq 1$, X r.v.s in E.

If $X_n \xrightarrow{P} X$ or $X_n \xrightarrow{a.s} X$ or $X_n \xrightarrow{L^p} X$, ($E = \mathbb{R}$.)

then $X_n \xrightarrow{(d)} X$.

Proof. Since a.s. cv and L^p-cv imply convergence in probability, it suffices to prove the result in the case $X_n \xrightarrow{P} X$.

Let $f : E \to \mathbb{R}$ continuous bounded. We have $f(X_n) \xrightarrow{P} f(X)$.

Let $\varepsilon > 0$. We have

$$\left| E\big(f(X_n) - f(X)\big)\right| \leq E\Big(\underbrace{|f(X_n) - f(X)|}_{\leq \varepsilon} 1_{|f(X_n)-f(X)| \leq \varepsilon}\Big)$$

$$+ E\Big(\underbrace{|f(X_n) - f(X)|}_{\leq 2\|f\|_\infty} 1_{|f(X_n)-f(X)| > \varepsilon}\Big)$$

$$\leq \varepsilon + 2\|f\|_\infty P\big(|f(X_n) - f(X)| > \varepsilon\big)$$

Hence $\limsup_{n\to\infty} \left| E\big(f(X_n) - f(X)\big)\right| \leq \varepsilon$. ∎

Interpretation: Convergence in probability is "stronger" than convergence in distribution in the following sense:

$X_n \xrightarrow{P} X \iff X_n \xrightarrow{(d)} X$ and X_n and X are constructed in such a way that they are spatially close to each other.

The reciprocal is wrong in general.

$X \sim Ber\left(\frac{1}{2}\right) \quad X_n = 1 - X$.

Application 1

$E = \mathbb{R} \quad X_1, X_2, \dots$ iid in $L^1 \quad m = E(X_1)$.

$$\frac{X_1 + \dots + X_n}{n} \xrightarrow{(d)} m.$$

Application 2 (alternative proof of $\mathcal{U}(\{0, \frac{1}{n}, \dots, 1\}) \to \mathcal{U}([0,1])$.

Let $U \sim \mathcal{U}([0,1])$.

For $n \geq 1$ let $X_n = \dfrac{\lfloor (n+1) U \rfloor}{n}$ $\left(X_n \sim \mathcal{U}(\{0, \frac{1}{n}, \dots, 1\}) \right)$.

$X_n \xrightarrow{a.s.} U$, hence $X_n \xrightarrow{(d)} U$.

Rk: A reciprocal result exists (Skorokhod representation)

If $\mu_n \xrightarrow{(w)} \mu$, then $\exists\, (\Omega, \mathcal{F}, P)$ $\exists\, X_n, X$ r.v. on $(\Omega, \mathcal{F}, P)$ s.t. $\mu_{X_n} = \mu_n$ $\mu_X = \mu$ and $X_n \xrightarrow{a.s.} X$

(see exercises for $E = \mathbb{R}$)

6 CONSTANT LIMITS.

Prop. Let $X_n, n \geq 1$ rvs on E. $x \in E$ constant.

$$X_n \xrightarrow{P} x \iff X_n \xrightarrow{(d)} x.$$

Proof. $\Rightarrow$ ok

$\Leftarrow$ Let $\varepsilon > 0$. We have, for $n \geq 1$,

$$P(d(X_n, x) \geq \varepsilon) = P(X_n \in E \setminus B(x, \varepsilon))$$

Since $E \setminus B(x, \varepsilon)$ closed, by Portmanteau theorem we have

$$\limsup_{n \to \infty} P(d(X_n, x) \geq \varepsilon) \leq P(x \in E \setminus B(x, \varepsilon)) = 0 \blacksquare$$

7 SLUTSKY THEOREM.

$E = \mathbb{R}^d$

Thm (Slutsky):

$X_n, n \geq 1$, X r.v.s in $\mathbb{R}^d$

$Y_n, n \geq 1$ random variables in $\mathbb{R}^{d'}$ $c \in \mathbb{R}^{d'}$ cst

$$\left.\begin{array}{l} X_n \xrightarrow{(d)} X \\ Y_n \xrightarrow{(d)} c \end{array}\right\} \Longrightarrow (X_n, Y_n) \xrightarrow{(d)} (X, c)$$

Proof Let $F \subset \mathbb{R}^{d+d'}$ closed set. For $k \geq 1$ we have

$$P((X_n, Y_n) \in F) = P((X_n, Y_n) \in F, \|Y_n - c\| \leq \frac{1}{k}) + P((X_n, Y_n) \in F, \|Y_n - c\| > \frac{1}{k})$$

$$\leq P((X_n, c) \in F^{1/k}) + P(\|Y_n - c\| \geq \frac{1}{k}),$$

where $F^{1/k} = \{ z \in \mathbb{R}^{d+d'} : d_\infty(z, F) \leq \frac{1}{k} \}$ closed.

Hence $\limsup_{n\to\infty} P((X_n, Y_n) \in F) \leq P((X, c) \in F^{1/k})$

Since $\bigcap_{k \geq 1} F^{1/k} = F$, by taking the limit as $k \to \infty$

we get $\limsup_{n\to\infty} P((X_n, Y_n) \in F) \leq P((X, c) \in F$ ∎

Corollary: X_n, X, Y_n r.v in $\mathbb{R}$ $X_n \xrightarrow{(d)} X$, $Y_n \xrightarrow{(d)} c \Rightarrow X_n Y_n \xrightarrow{(d)} Xc$

ii APPLICATION : NORMALITY OF t - STATISTIC .

Let $X_1, X_2, \ldots$ iid real r.v.s in L^2

$m = E(X_1)$ $\sigma^2 = Var(X_1)$

$$\overline{X}_n = \frac{X_1 + \cdots + X_n}{n} \qquad S_n^2 = \frac{1}{n-1}\sum_{k=1}^{n} (X_k - \overline{X}_n)^2 = \frac{1}{n-1}\sum_{k=1}^{n} X_k^2 - \frac{n}{n-1}\overline{X}_n^2$$

$$\frac{X_1 + \cdots + X_n - nm}{\sqrt{n S_n^2}} \xrightarrow{(d)} Z \text{ where } Z \sim \mathcal{N}(0,1)$$

Proof: By the law of large numbers applied to (X_i) and (X_i^2) we have

$$\overline{X}_n \xrightarrow{a.s} E(X_1)$$

$$\frac{X_1^2 + \cdots + X_n^2}{n} \xrightarrow{a.s.} E(X_1^2)$$

Hence, by operation,

$$S_n^2 \xrightarrow{a.s.} E(X_1^2) - E(X_1)^2 = \sigma^2.$$

By the central limit theorem

$$Z_n := \frac{X_1 + \cdots + X_n - n \cdot m}{\sqrt{n\sigma^2}} \xrightarrow{(d)} Z \sim \mathcal{N}(0,1)$$

Finally $T_n = Z_n \times \sqrt{\frac{\sigma^2}{S_n^2}}$

Since $Z_n \xrightarrow{(d)} Z$ and $\sqrt{\frac{\sigma^2}{S_n^2}} \xrightarrow{a.s.} 1$, by Slutsky Thm, we have

$$T_n \xrightarrow{(d)} Z \times 1$$