# INRODUCTION : A WALK IN ZÜRICHBERG

Starting point

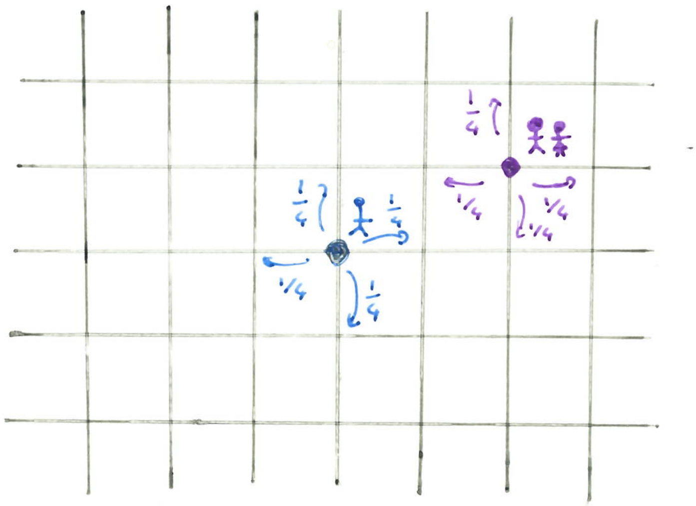
Zürichberg Hotel, 1918.

George Pólya (1887 - 1985). Hungarian mathematician.

ETH Professor (1914-1940)

Model for the walk in the wood.

Woods $= \mathbb{Z}^2$.




Will Polya and the student meet infinitely many time ?

# 1 POLYA'S THEOREM (1921)

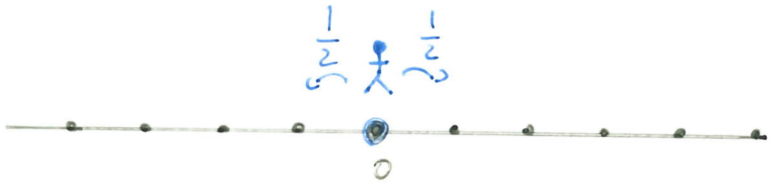
Reformulation of the problem :

Only 1 walker starting at 0.

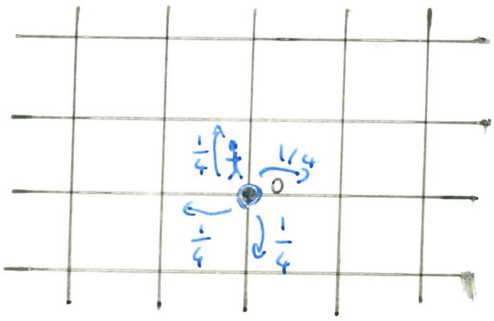
Does the walker visit 0 infinitely often?

Generalization : $\mathbb{Z}^d$, $d \geq 1$ arbitrary dimensions

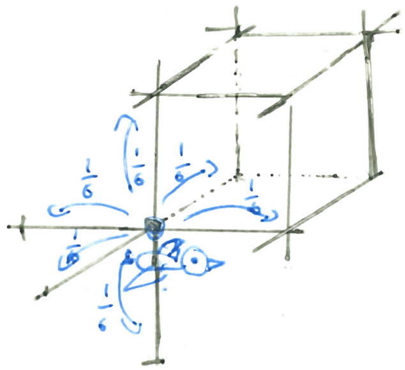
$\mathbb{Z}$ =




$\mathbb{Z}^2$




$\mathbb{Z}^3$



$\mathbb{Z}^d$ $\rightarrow$ jump probability = $\dfrac{1}{2d}$

**Thm (Polya '1921)**

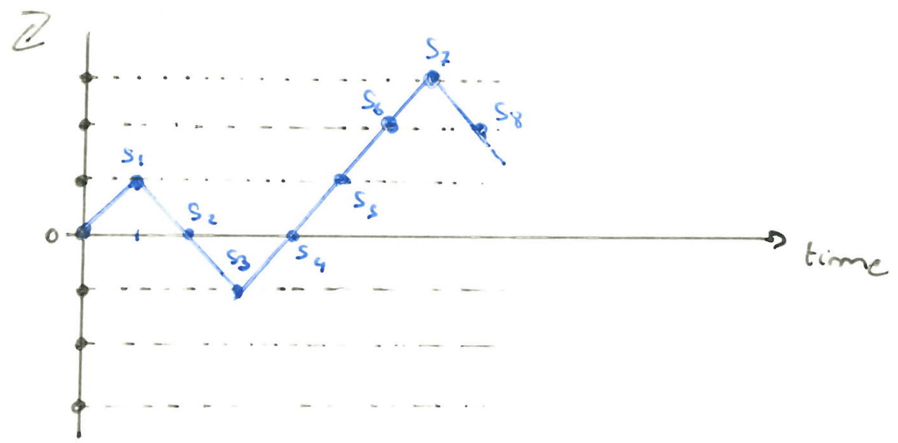
$$\boxed{\begin{array}{l} d = 1, 2 : \quad P\left(\begin{array}{c}\text{walker visits } 0\\ \text{infinitely often}\end{array}\right) = 1. \\ \\ d \geq 3 : \quad P\left(\begin{array}{c}\text{walker visits } 0\\ \text{finitely often}\end{array}\right) = 1. \end{array}}$$

## 2 FORMAL DEFINITION ON $\mathbb{Z}$

$X_1, X_2, X_3, \ldots$ iid random variables

$$P(X_i = -1) = P(X_i = +1) = \frac{1}{2}$$

$S_n = X_1 + \cdots + X_n$ Position of the walker after $n$ steps.

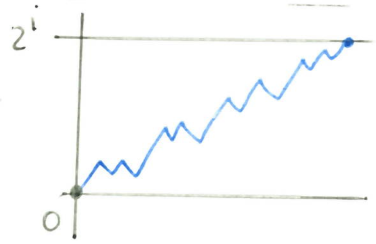


## Proof idea of Polya's Theorem $(d=1)$

Goal: $P(\exists n \geq 1 : S_n = 0) = 1$

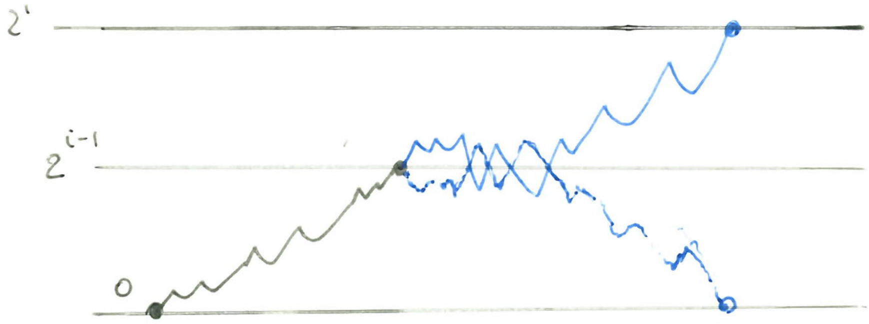
Define for $i \geq 0$

$A_i =$




"The walker reaches $2^i$ before returning to 0"

Reflection trick:

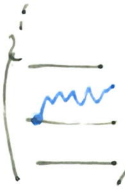
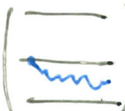



$\forall i \geq 1 \quad P(A_i \mid A_{i-1}) \leq \frac{1}{2}$

Hence
$$P(A_i) = P(A_i \cap A_{i-1}) = P(A_i \mid A_{i-1}) \times P(A_{i-1})$$
$$\leq \frac{1}{2} P(A_{i-1})$$

By induction $P(A_i) \xrightarrow[i \to \infty]{} 0$

Conclusion

$P(\forall n \geq 1 \; S_n \neq 0) \leq P\Big($  $\Big) + P\Big($  $\Big) + P\Big($  $\Big) + P\Big($  $\Big)$

| "reaches $2^i$ before 0" | "stays in $(0, 2^i)$" | "reaches $-2^i$ before 0" | "stays in $(-2^i, 0)$" |
|---|---|---|---|
| $\xrightarrow[i\to\infty]{} 0$ | $= 0$ (Borel-Cantelli) | $\xrightarrow[i\to\infty]{} 0$ (sym.) | $= 0$ |

# 3 RETURN PROBABILITY.

**d=1** $P(S_n = 0)$?

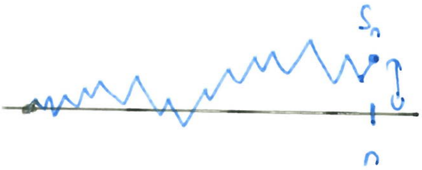
**Rk.** $\forall n \quad P(S_{2n+1} = 0) = 0$.

**Thm:** $$\boxed{P(S_{2n} = 0) \underset{n\to\infty}{\sim} \frac{1}{\sqrt{\pi n}}}.$$

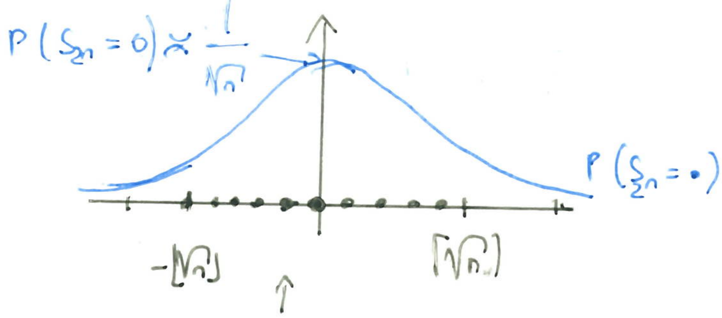
Is $\frac{1}{\sqrt{n}}$ surprizing?

$$\frac{S_n}{\sqrt{n}} \xrightarrow{(d)} \mathcal{N}(0,1)$$

(CLT)




$\simeq \sqrt{n} \times \mathcal{N}(0,1)$




$$\sum_{|i| \leq \sqrt{n}} P(S_{2n}=i) \asymp 1$$

## "Proof" of Thm

- $P(S_0 = 0) = 1$.
- $P(S_2 = 0) = P(\;) + P(\;) = 2 \times \frac{1}{4}$.
- $P(S_4 = 0) = P(\;) + P(\;) + P(\;) + P(\;) + P(\;) + P(\;)$

$$= 6 \times \frac{1}{16}.$$

- $P(S_{2n} = 0) = \#\{\underbrace{\qquad\qquad}_{2n}\} \times \frac{1}{2^{2n}}$

$$= \binom{2n}{n} \cdot \frac{1}{2^{2n}}.$$

Stirling : $n! \underset{n\to\infty}{\sim} \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$.

$$P(S_{2n}=0) = \frac{(2n)!}{(n!)^2} \times \frac{1}{2^{2n}}$$

$$\underset{n\to\infty}{\sim} \frac{\sqrt{4\pi n}\left(\frac{2n}{e}\right)^{2n}}{2\pi n\left(\frac{n}{e}\right)^{2n}} \times \frac{1}{2^{2n}} = \frac{1}{\sqrt{\pi n}} \qquad \blacksquare$$

## 4 AND IN HIGHER DIMENSIONS ?

$X_i \in \{\pm e_j\}_{j=1}^{d}$ $\qquad e_j = (0,\dots 0, \underset{\substack{\uparrow\\ j}}{1}, 0 \dots 0)$

$S_n = X_1 + \dots + X_n \in \mathbb{Z}^d$ $\qquad S_n = (S_n^{(1)}, \dots, S_n^{(d)})$

- $P(S_{2n}=0) = P\left(\forall i \in \{1,\dots,d\}\ \ S_{2n}^{(i)}=0\right)$

$$\simeq \prod_{i=1}^{d} P(S_{2n}^{(i)}=0)$$

$$\simeq \left(\frac{1}{\sqrt{n}}\right)^d .$$

$$A_n = \{S_{2n} = 0\}$$

The walker visits 0 infinitely many times $\Longleftrightarrow$ Infinitely many $A_n$ occur

Borel Cantelli $\Rightarrow$

$\Leftarrow$ Markov Chain Theory

$$\Longleftrightarrow \sum_{n \geq 1} \underbrace{P(A_n)}_{\simeq \frac{1}{n^{d/2}}} = \infty$$

$$\Longleftrightarrow \frac{d}{2} \leq 1$$

# REMINDERS

## 1 PROBABILITY SPACE.

**Def.** A probability space is a triple $(\Omega, \mathcal{F}, P)$, where

- $\Omega$ is a non-empty set.
- $\mathcal{F}$ is a $\sigma$-algebra on $\Omega$.

  [i.e. $\mathcal{F} \subset \mathcal{P}(\Omega)$ and satisfies:
  - $\emptyset \in \mathcal{F}$,
  - $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$,
  - $A_1, A_2, \dots \in \mathcal{F} \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$.

- $P$ is a probability measure on $(\Omega, \mathcal{F})$.

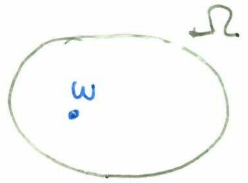
  [i.e. $P : \mathcal{F} \to [0,1]$ and satisfies
  - $P(\Omega) = 1$,
  - $\forall A_1, A_2, \dots \in \mathcal{F}$ pairwise disjoint

$$P\left(\bigsqcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P(A_n).$$

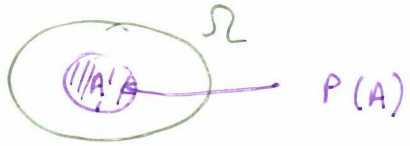
**Prop.** If $\mathcal{F}$ is a $\sigma$-algebra, then

- $\Omega \in \mathcal{F}$
- $\forall A_1, A_2, \dots \in \mathcal{F}$ $\quad \bigcap_{n \geq 1} A_n \in \mathcal{F}$.

Intuition: We think of an element $\omega \in \Omega$ as randomly chosen in $\Omega$. (the output of a random experiment).




For $A \subset \Omega$, $P(A)$ represents the probability that this random element lies in $A$




Ex 1: Die throw

$\Omega = \{1, 2, 3, 4, 5, 6\}$

$\mathcal{F} = \mathcal{P}(\Omega)$

$P(A) = \dfrac{|A|}{6}$ for every $A \in \mathcal{F}$.

Ex 2 : $\Omega = [0,1]$

$\mathcal{F} = \mathcal{B}([0,1])$ Borel $\sigma$-algebra.

$P = \mathcal{L}$ Lebesgue measure.

Elements of $\mathcal{F}$ are called events. They can be thought as properties of the output. For instance, in Ex 1,

$A = \{1, 3, 5\}$ "the die is odd"

In Ex 2

$A = [0, \frac{1}{2}] \setminus \mathbb{Q}$ "the random point is $\leq \frac{1}{2}$ and irrational"

From now, we fix a probability space $(\Omega, \mathcal{F}, P)$.

Def. Let $A \subset E$. We say that $A$ occurs almost surely if $\exists A' \in \mathcal{F}$ s.t. $A' \subset A$ $P(A') = 1$

In particular, if $A \in \mathcal{F}$ $A$ occurs a.s. $\Leftrightarrow P(A) = 1$

Prop. Let $A_1, A_2, \dots \subset E$.

"countable intersection"

$(\forall i \geq 1 \; A_i \text{ occurs a.s.}) \Rightarrow (\bigcap_{i=1}^{\infty} A_i \text{ occurs a.s.})$

## BOREL - CANTELLI THEOREMS

$A_1, A_2, \dots \in \mathcal{F}$ sequence of events.

$$\limsup A_n := \bigcap_{N \geq 1} \bigcup_{n \geq N} A_n \quad \text{"infinitely many } A_n \text{ occur"}$$

$\boxed{\text{I}}$ If $\sum_{n \geq 1} P(A_n) < \infty$, then $P(\limsup A_n) = 0$.

"a.s. finitely many $A_n$ occur"

$\boxed{\text{II}}$ If $\sum_{n \geq 1} P(A_n) < \infty$ and $A_1, A_2, \dots$ are independent, then $P(\limsup A_n) = 1$.

"a.s. infinitely many $A_n$ occur"

## 2 RANDOM VARIABLES

$(E, \mathcal{E})$ fixed measurable space.

- often $E = \mathbb{R}$ $\mathcal{E} = \mathcal{B}(\mathbb{R})$
- $E$ topological space $\mathcal{E}$ = borel $\sigma$-algebra.

**Def.** A random variable with values in $E$ is a measurable map $X : \Omega \longrightarrow E$.

$$X \text{ measurable} \iff \forall B \in \mathcal{E} \;\; X^{-1}(B) \in \mathcal{F}.$$

$$\iff \forall C \in \mathcal{C} \;\; X^{-1}(C) \in \mathcal{F}$$

if $\mathcal{E} = \sigma(\mathcal{C})$

Intuition 1:

$\omega \in \Omega$
random experiment $\longrightarrow$ $X(\omega)$ observation

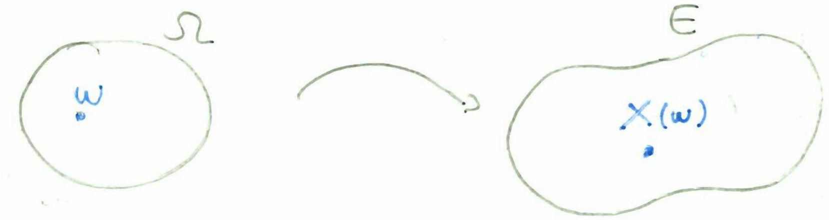
Ex: $\Omega = \{1, 2, 3, 4, 5, 6\}$

$$X(\omega) = \begin{cases} 0 & \text{if } \omega \in \{1,3,5\} \\ 1 & \text{if } \omega \in \{2,4,6\} \end{cases}$$

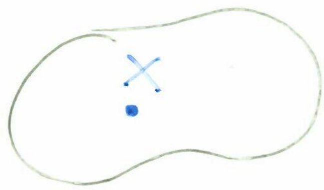
"X observes if $\omega$ is even".

Intuition 2. (the one that we use mostly in this course.)

Formally: $X$ is a function $: \Omega \to E$




Intuitively: $X$ is a random point in $E$




Notation: • for $B \in \mathcal{E}$ $\quad \{X \in B\} := \{\omega \in \Omega : X(\omega) \in B\}$

• for $x \in E$ $\quad \{X = x\} := \{\omega \in \Omega : X(\omega) = x\}$.

...

With this intuition, the precise construction of $X$ (with an explicit $\Omega$) is not really important. Its probabilistic features (distribution, density ...), its relationship to other random variables (independence ...) is more central.

For example, we say that a random variable is uniform in $\{1, 2, 3\}$ if

$$P(X = 1) = P(X = 2) = P(X = 3) = \frac{1}{3}. \quad (*)$$

There are several ways to construct such a random variable. For example

- $X(\omega) = \lfloor \frac{\omega+1}{2} \rfloor$ if $\Omega = \{1,2,3,4,5,6\}$, $P = \frac{1 \cdot 1}{6}$
- $X(\omega) = 1 + \lfloor 3\omega \rfloor$ if $\Omega = [0,1]$, $P = \mathcal{L}$.

The properties of $X$, given by the equation (*) are generally more important than the way $X$ is constructed.

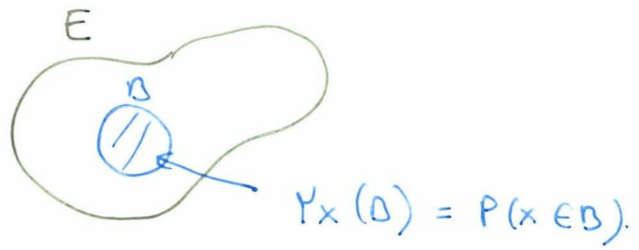
In probability theory, we generally "omit" the probability space that we are working on, and focus on the properties of the random variables that we are considering. This allows us to consider random variables as points. For instance, we may say: "Let $X \in \{1,2,3\}$ be uniformly chosen" instead of "Let $X : \Omega \to \{1,2,3\}$ be a uniform random variable"

$\hookrightarrow$ $\Omega$ is fixed and implicit.

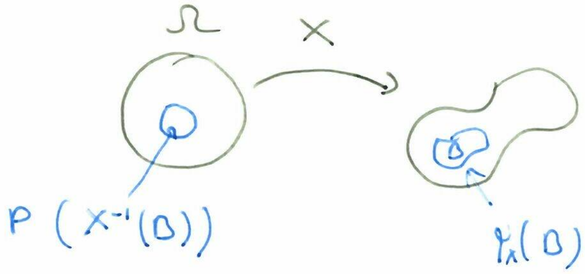
## Law of a r.v. $X$:

$\mu_X$ probability measure on $(E, \mathcal{E})$ def. by

$$\boxed{\forall B \in \mathcal{E} \quad \mu_X(B) = P(\underbrace{X \in B}_{\{\omega \in \Omega :\, X(\omega) \in B\}})}$$




Rk: $\mu_X$ is the pushforward measure of $P$ by $X$: $\mu_X = X \# P$.




Two fundamental examples: $\quad p \in [0,1]$

$X \sim \mathrm{Ber}(p) \iff \mu_X = p\,\delta_1 + (1-p)\,\delta_0$


$\iff P(X=1) = p \quad P(X=0) = 1-p$.

$X \sim \mathcal{U}([0,1]) \iff \mu_X = \mathcal{L}_{[0,1]}$ "Lebesgue measure"

$\iff \forall\, 0 \le a \le b \le 1 \quad P(a \le X \le b) = b - a$

Construction of r.v.

[mapping] $(E_1, \mathcal{E}_1), \dots, (E_n, \mathcal{E}_n)$ measurable spaces

$\varphi : E_1 \times \cdots \times E_n \longrightarrow E$ measurable

$\varphi(X_1, \dots, X_n) : \omega \mapsto \varphi(X_1(\omega), \dots, X_n(\omega))$ r.v.

[limit] $X_1, X_2, \dots$ real r.v.

$\limsup X_n$ and $\liminf X_n$ are r.v.

## 3 EXPECTATION. (real r.v. $X : \Omega \to \mathbb{R}$)

$\{X \text{ r.v. } \int |X| \, dP < \infty\}$

**Def.** Let $X \geq 0$ a.s. or $X \in L^1(P)$, we define

$$\boxed{E(X) = \int_\Omega X \, dP.}$$

Intuition: "typical value taken by $X$"

Rk: if $\Omega$ finite or countable

$$E(X) = \sum_{\omega \in \Omega} X(\omega) . P(\{\omega\})$$

**Thm** (dominated convergence)

$X_1, X_2, \dots, X$ real r.v.

If ① $\lim_{n\to\infty} X_n = X$ a.s. (i.e. $P(\{\omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\}) = 1$)

② $\exists Z \in L^1(P)$ $\forall n$ $|X_n| \leq Z$ a.s.

Then $\lim_{n\to\infty} E(X_n) = E(X)$

**Thm** (linearity)

Let $X, Y \in L^1(P)$ $\alpha, \beta \in \mathbb{R}$.

$$E(\alpha X + \beta Y) = \alpha E(X) + \beta E(Y).$$

**Prop.** (Markov inequality)

Let $X$ r.v. $X \geq 0$ a.s., $a > 0$.

$$\boxed{P(X \geq a) \leq \frac{1}{a} E(X)}$$